Bayesian Nonparametrics

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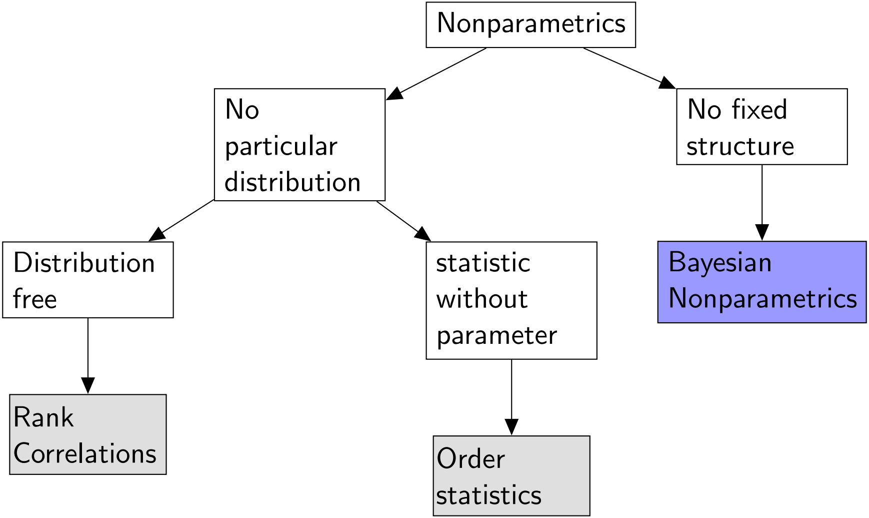
Nonparametrics



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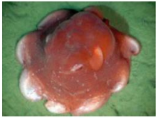
Nonparametric

no *finite* parameter. Allows for unbounded, growing, infinite number of parameters

Motivation

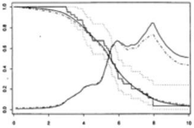
Practical and Theoretical

[wikipedia.org]

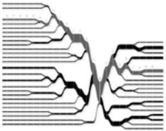


[Ed Bowlby, NOAA]

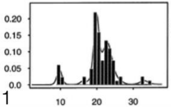
[Fox et al 2014]



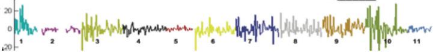
[Arjas, Gasbarra 1994]



[Ewens 1972; Hartl, Clark 2003]



[Escobar, West 1995; Ghosal et al 1999]



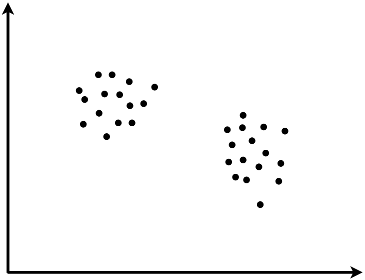
[Saria et al 2010]

DeFinetti's theorem

A sequence is infinitely exchangeable (distribution invariant to sequence) if and only if for all N and some distribution P:

$$p(X_1, \ldots, X_N) = \int_\theta \prod_{n=1}^{N} p(X_n|\theta)P(d\theta)$$

Generative model



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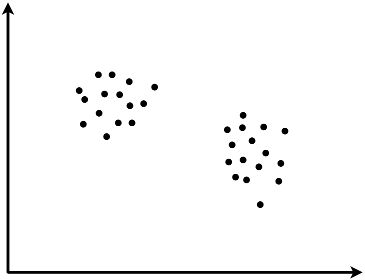

- Finite Gaussian mixture model (K=2 clusters)

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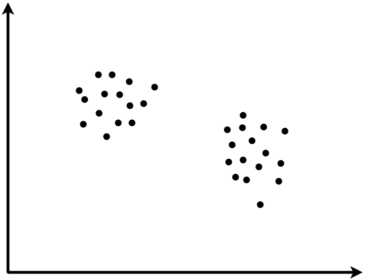

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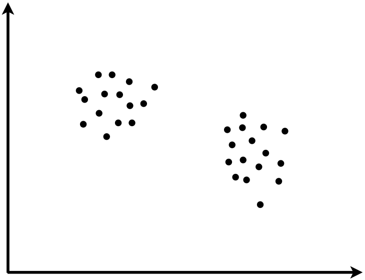

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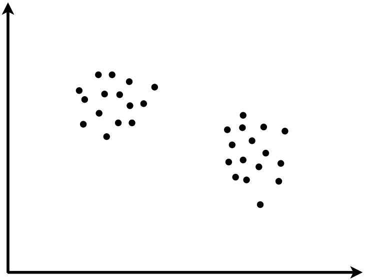

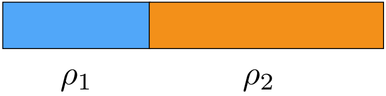

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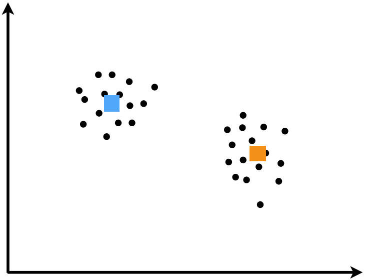

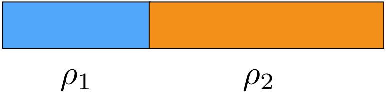

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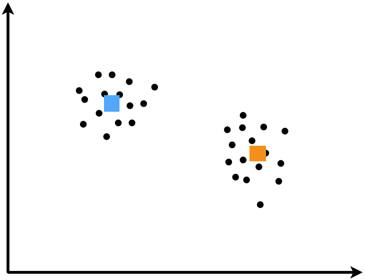

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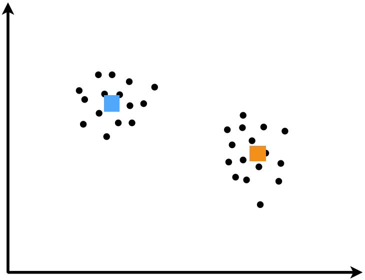

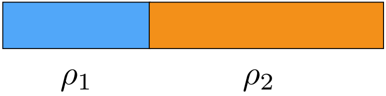

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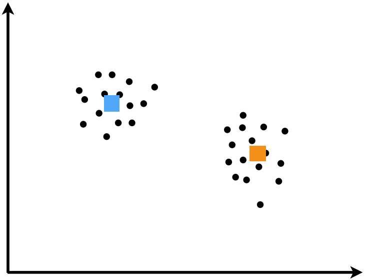

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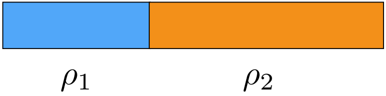

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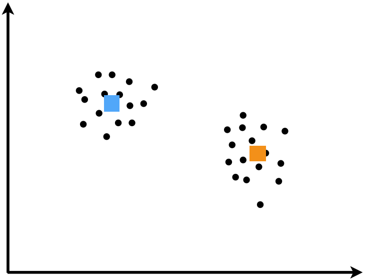

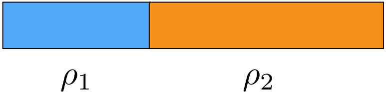

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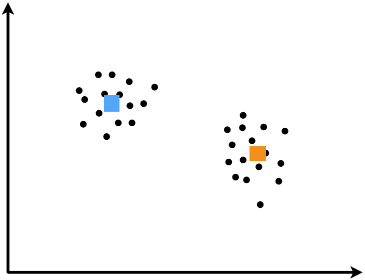

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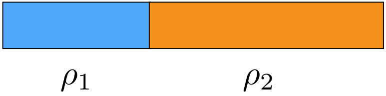

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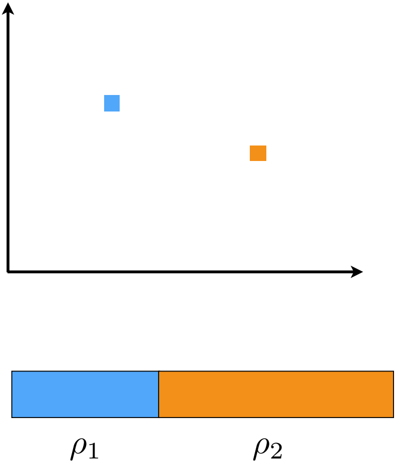

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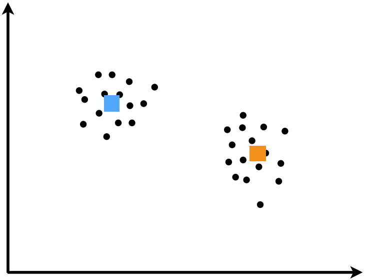

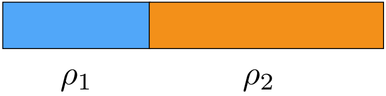

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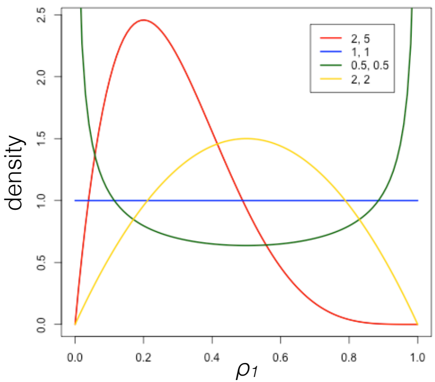

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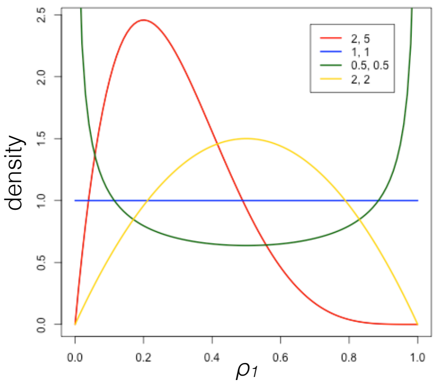

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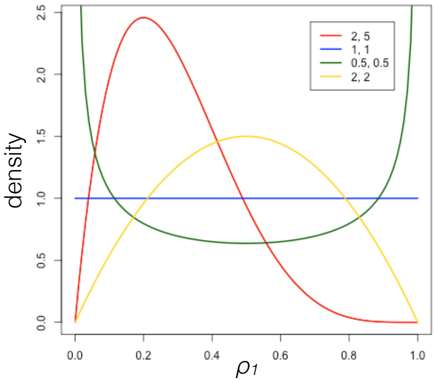

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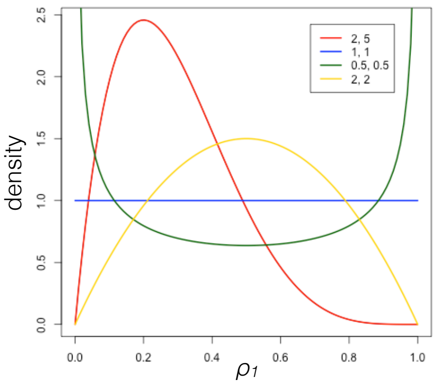

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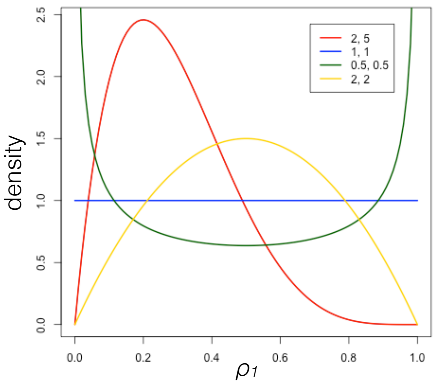

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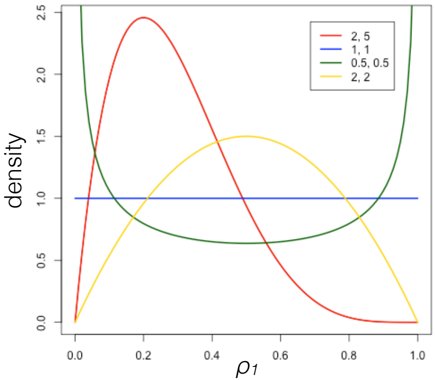

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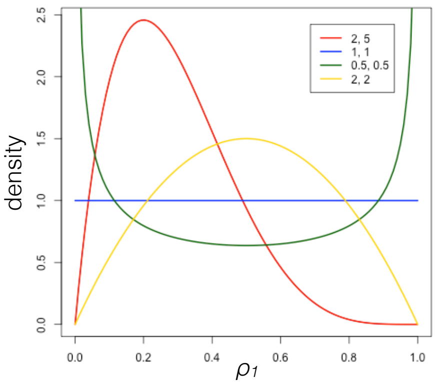

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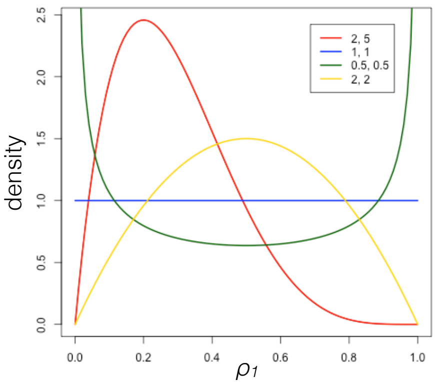

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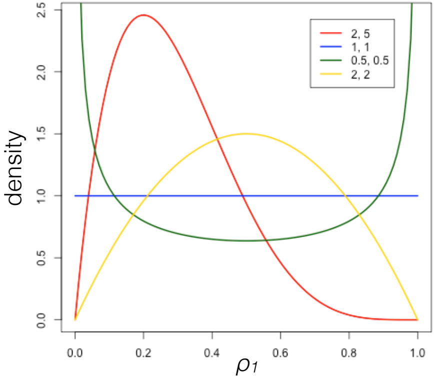

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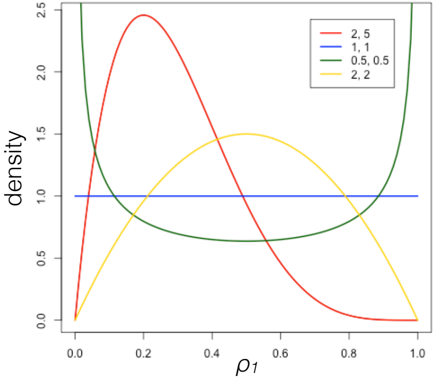

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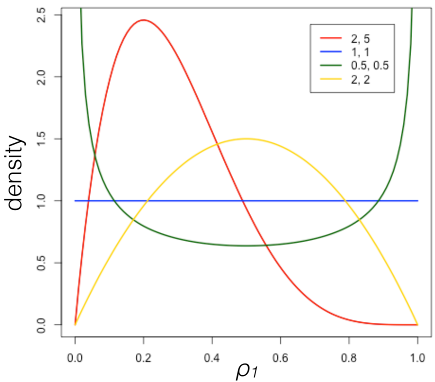

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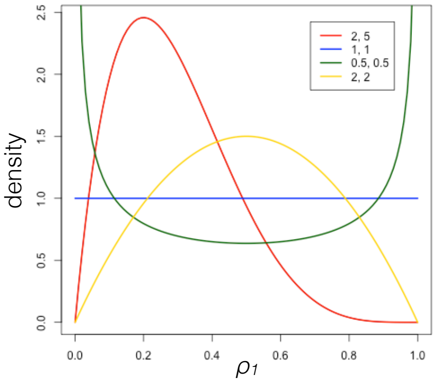

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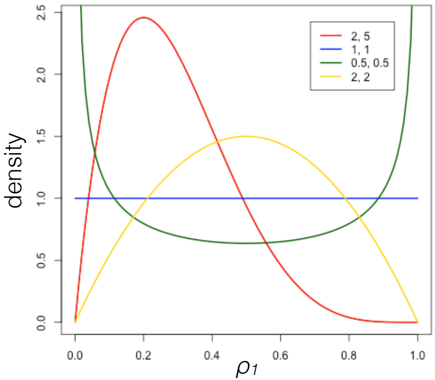

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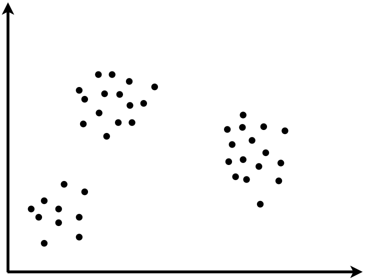

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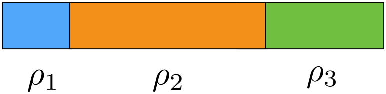

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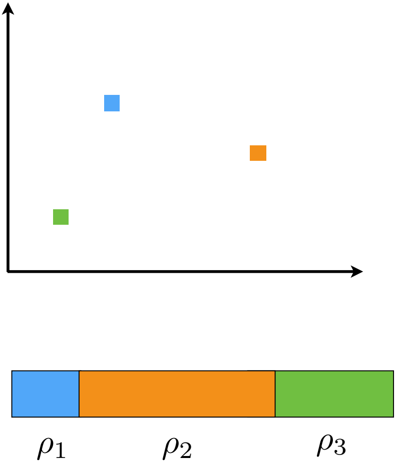

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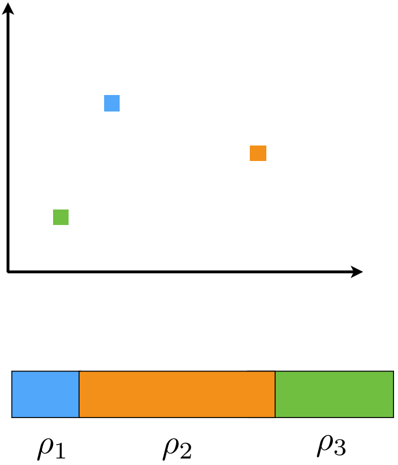

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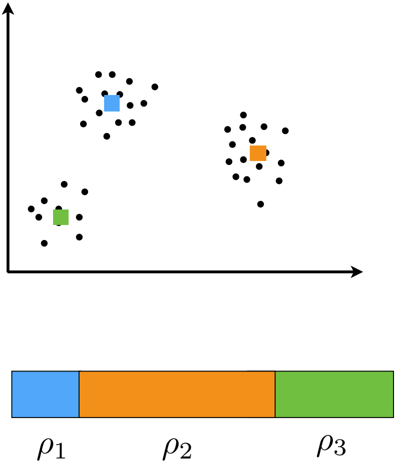

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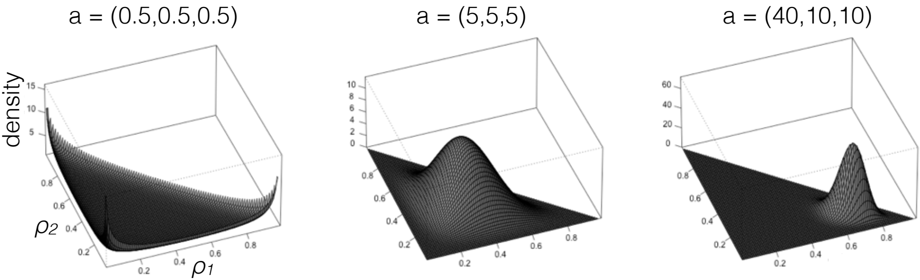

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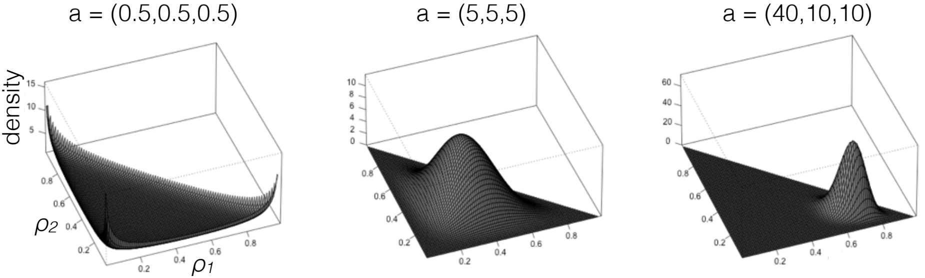

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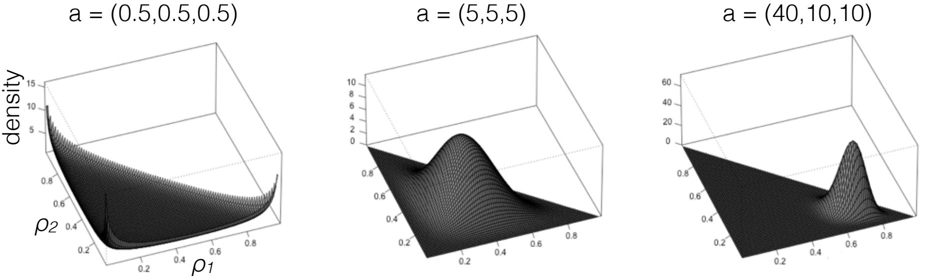

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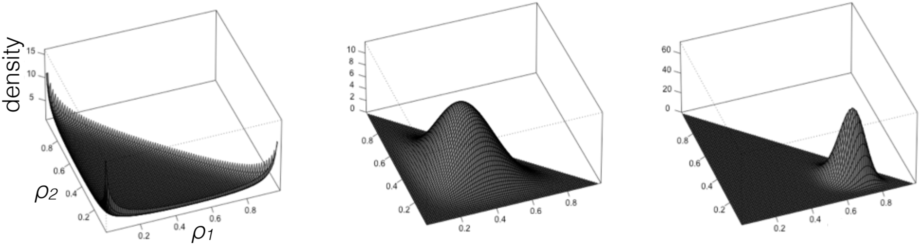
a = (0.5,0.5,0.5) a = (5,5,5) a = (40,10,10)



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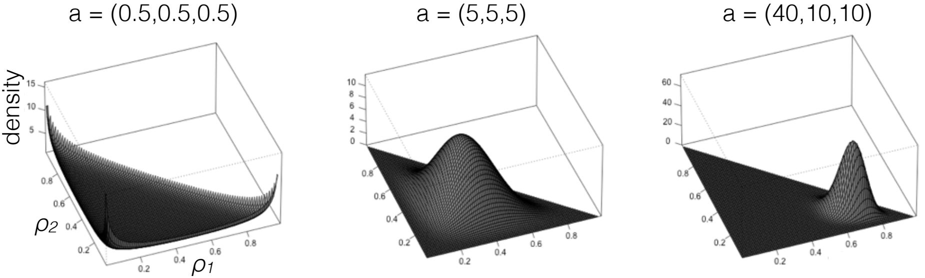

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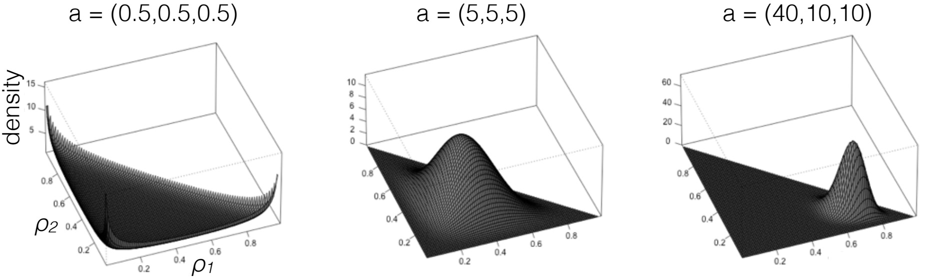

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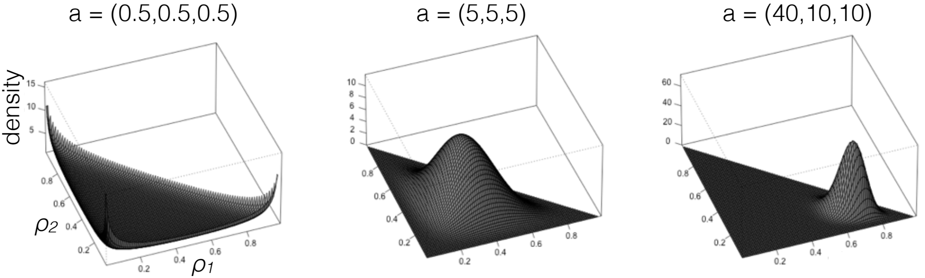

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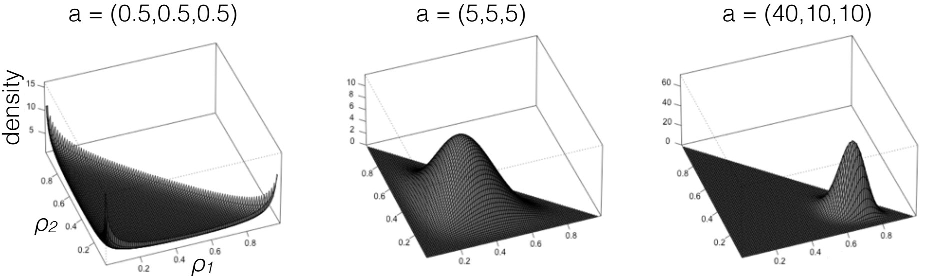

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How to extend to nonparametric?

- Beta is a prior on cluster probabilities for K=2
- Generalizes to Dirichlet for K=3
- Both are conjugate to Categorical

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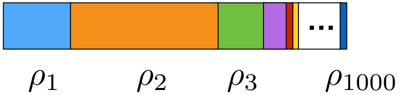

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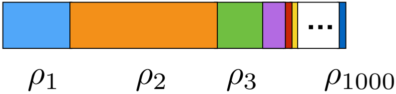

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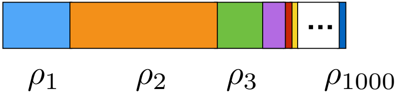

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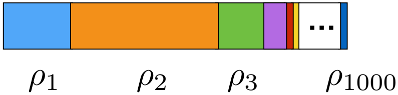

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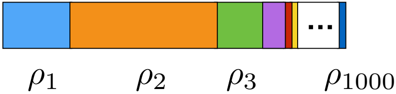

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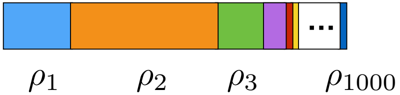

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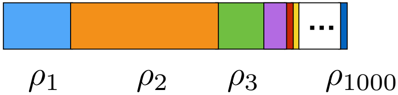

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- "Stick breaking"

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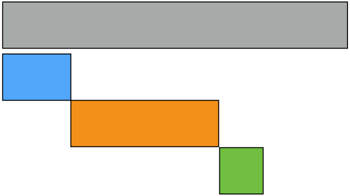

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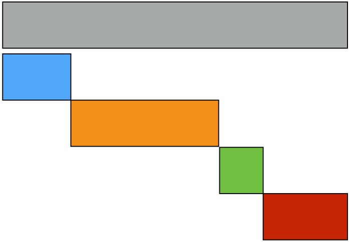

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$$\rho_4 = 1 - \sum_{k=1}^{3} \rho_k$$

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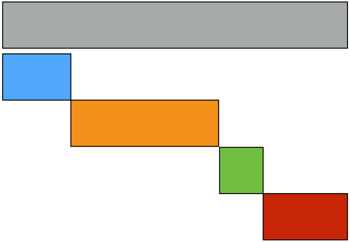

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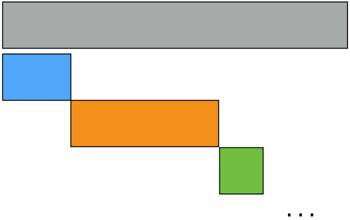

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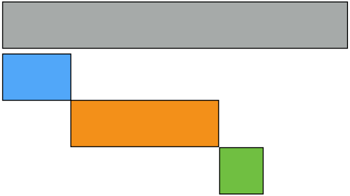

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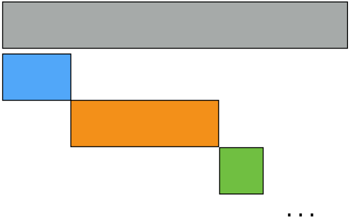

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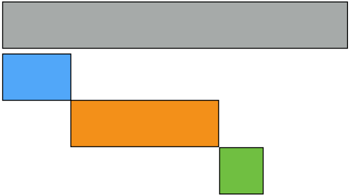

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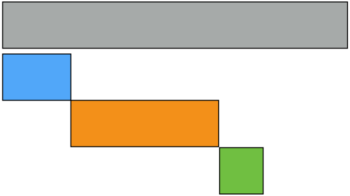

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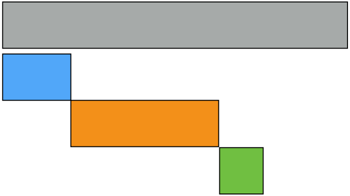

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[Ishwaran, James 2001]

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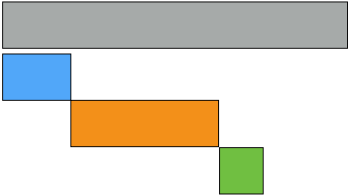

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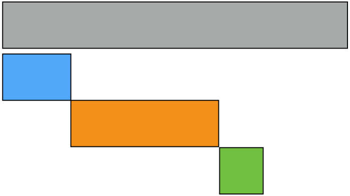
  - Griffiths-Engen-McCloskey (**GEM**) distribution:

$$\rho = (\rho_1, \rho_2, \ldots) \sim \mathrm{GEM}(\alpha)$$



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[McCloskey 1965; Engen 1975; Patil and Taillie 1977; Ewens 1987; Sethuraman 1994; Ishwaran, James 2001]

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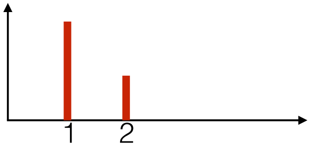

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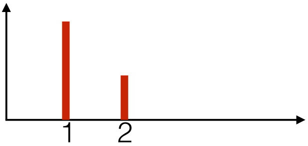

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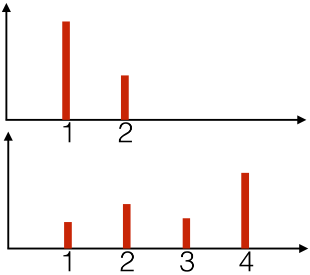

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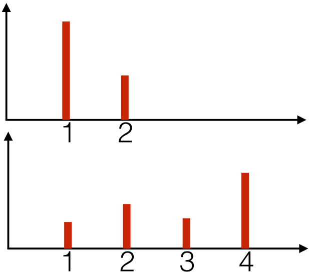

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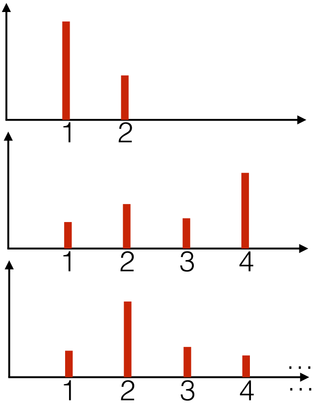

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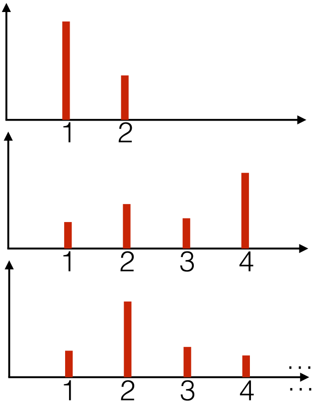

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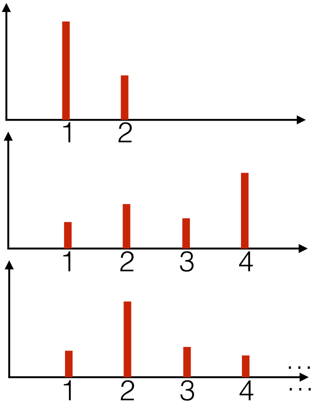

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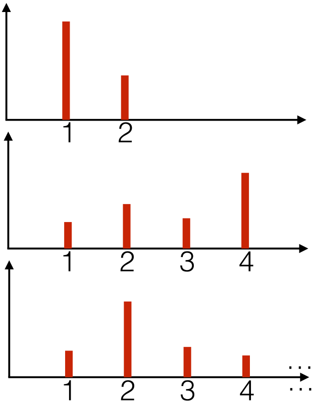

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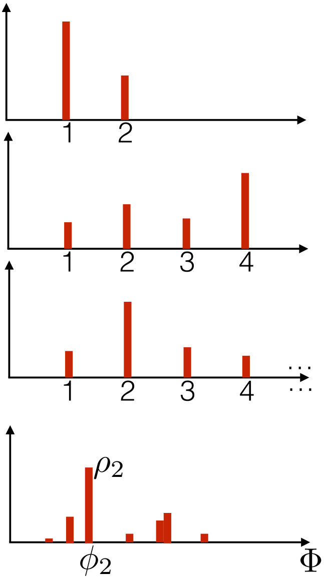

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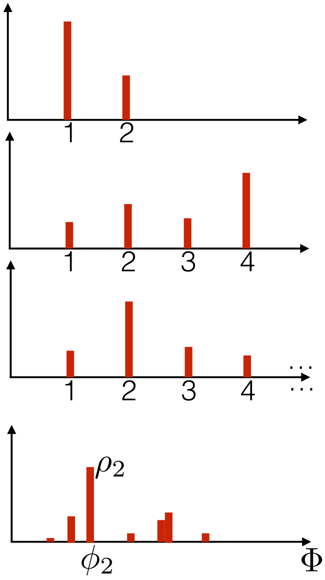

[Ferguson 1973]

Dirichlet process mixture model

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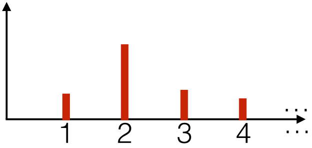

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$$\rho = (\rho_1, \rho_2, \ldots) \sim \mathrm{GEM}(\alpha)$$

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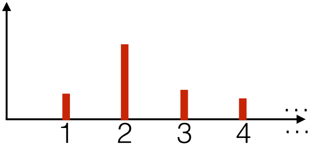

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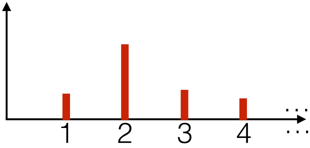

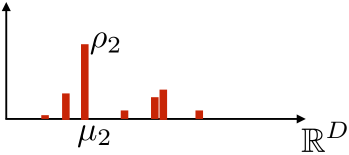

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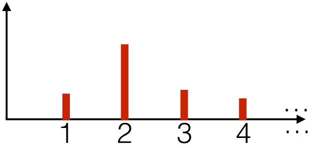
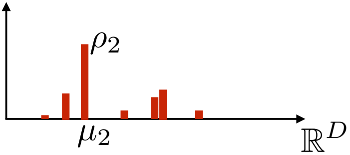
  - i.e. $G = \sum_{k=1}^{\infty} \rho_k \delta_{\mu_k}$



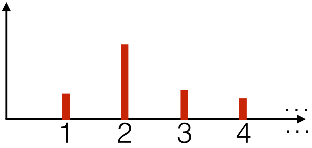


Dirichlet process mixture model

- Gaussian mixture model

$$\rho = (\rho_1, \rho_2, \ldots) \sim \text{GEM}(\alpha)$$

$$\mu_k \overset{iid}{\sim} \mathcal{N}(\mu_0, \Sigma_0), k = 1, 2, \ldots$$



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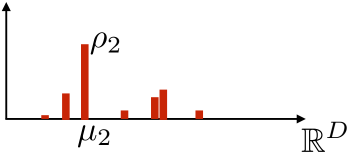

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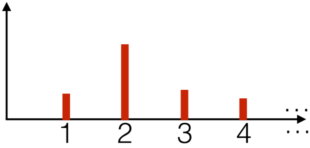

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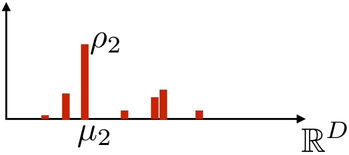

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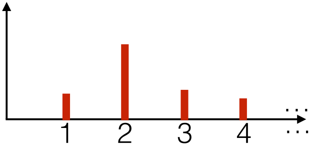

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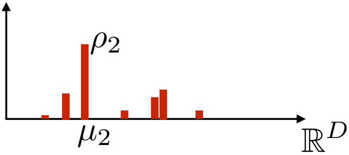

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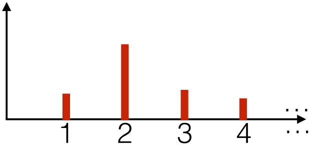

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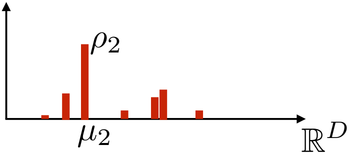

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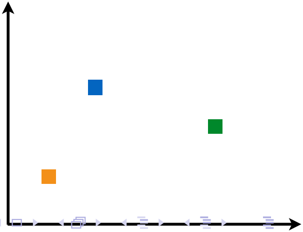

Dirichlet process mixture model

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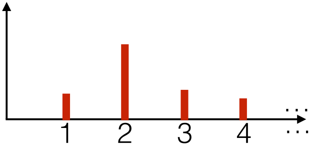

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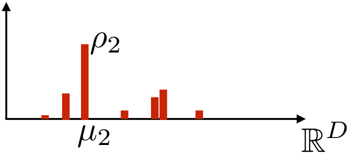

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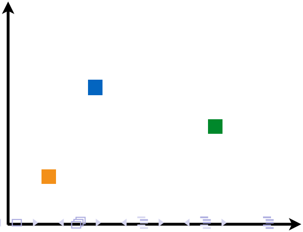

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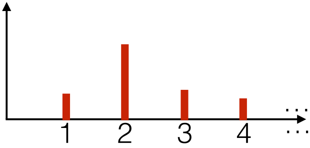

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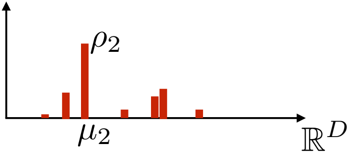

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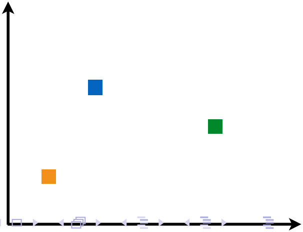

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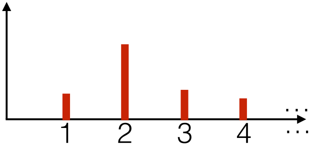

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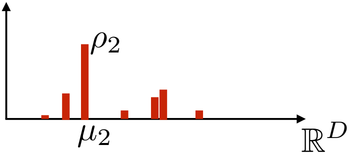

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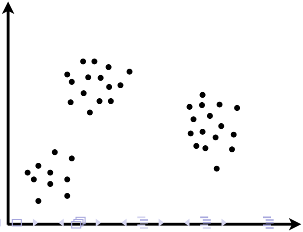

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Dirichlet process mixture model

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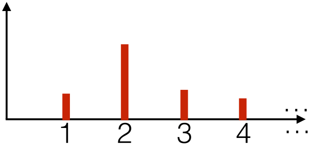

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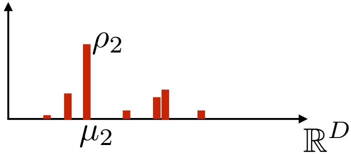

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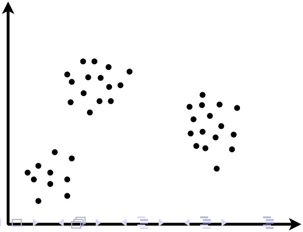

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[demo]

Dirichlet process mixture model

- More generally

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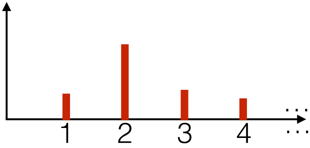

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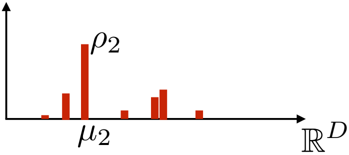

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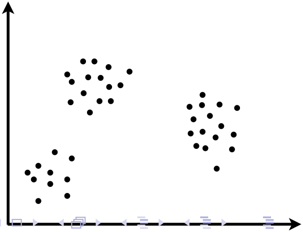

Dirichlet process mixture model

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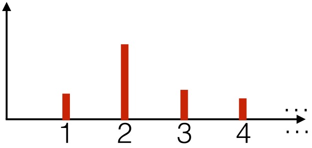

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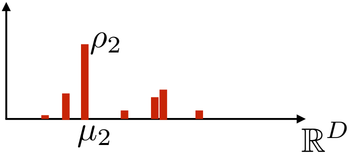

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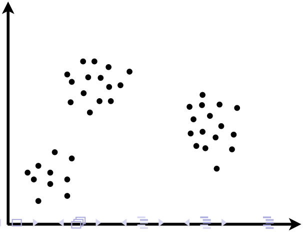

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Dirichlet process mixture model

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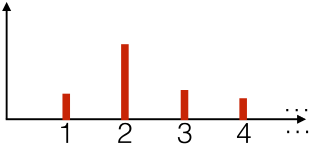

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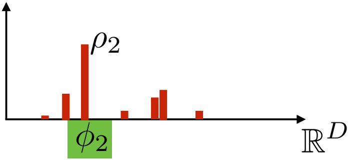

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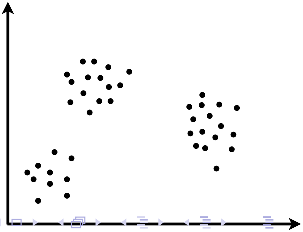

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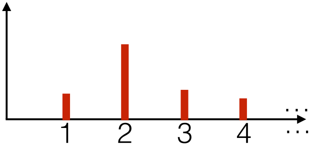

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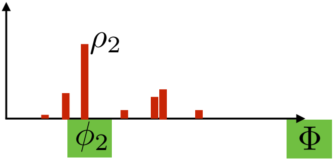

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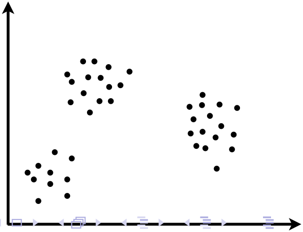

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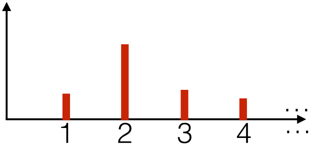

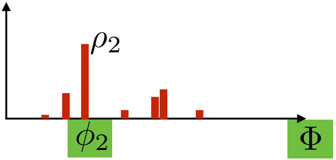

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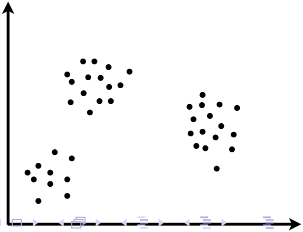

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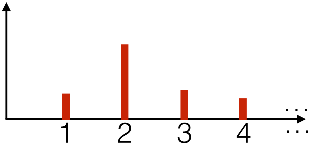

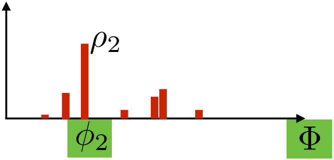

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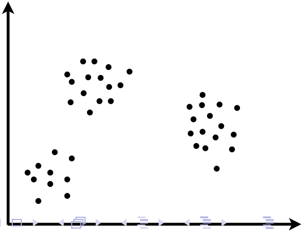

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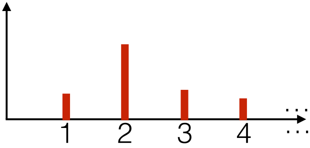

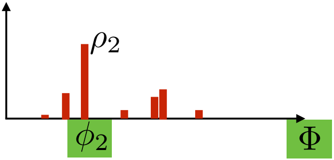

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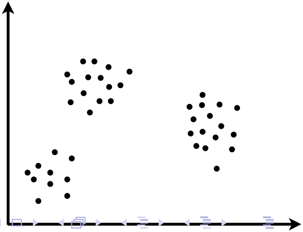

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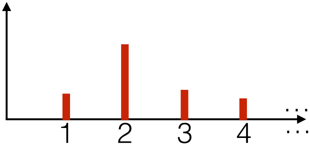

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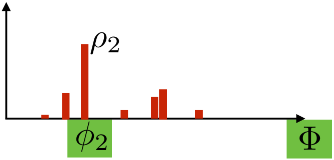

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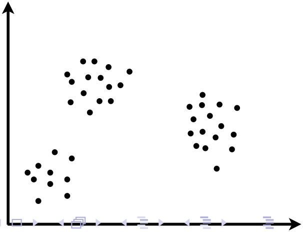

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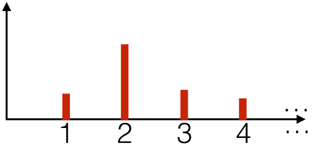
- More generally

$$\rho = (\rho_1, \rho_2, \ldots) \sim \text{GEM}(\alpha)$$

$$\phi_k \overset{iid}{\sim} G_0 \qquad k = 1, 2, \ldots$$

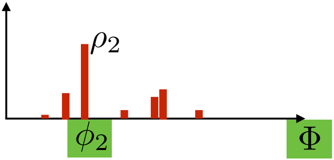
- i.e. $G = \sum_{k=1}^{\infty} \rho_k \delta_{\phi_k} \overset{d}{=} \text{DP}(\alpha, G_0)$



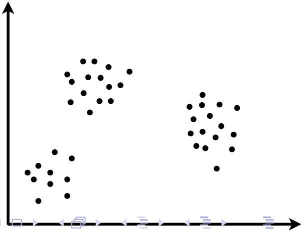
$$z_n \overset{iid}{\sim} \text{Categorical}(\rho)$$

$$\theta_n = \phi_{z_n}$$

- i.e. $\theta_n \overset{iid}{\sim} G$



$$x_n \overset{indep}{\sim} F(\theta_n)$$

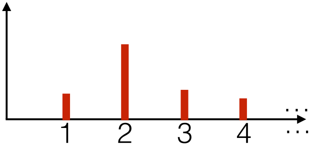


Dirichlet process mixture model

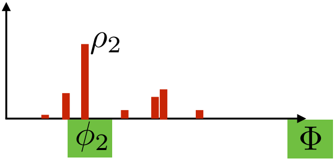
- More generally

$$\rho = (\rho_1, \rho_2, \ldots) \sim \mathrm{GEM}(\alpha)$$

$$\phi_k \overset{iid}{\sim} G_0 \qquad k = 1, 2, \ldots$$



- i.e. $G = \sum_{k=1}^{\infty} \rho_k \delta_{\phi_k} \overset{d}{=} \mathrm{DP}(\alpha, G_0)$

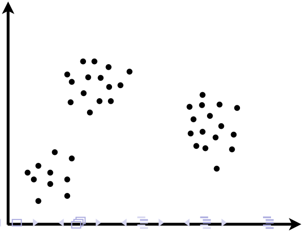


$$z_n \overset{iid}{\sim} \mathrm{Categorical}(\rho)$$

$$\theta_n = \phi_{z_n}$$

- i.e. $\theta_n \overset{iid}{\sim} G$

$$x_n \overset{indep}{\sim} F(\theta_n)$$



[Antoniak 1974; Ferguson 1983; West, Müller, Escobar 1994; Escobar, West 1995; MacEachern, Müller 1998]

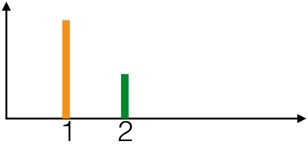
How can we use this?

- Generative model, infinite cluster prior
- Inference goal: cluster assignments, cluster parameters
- Even generating data: we can't sample infinity ρ's
- Marginals: $p(z_n | z_1, \ldots, z_{n-1})$

Again: consider 2 cluster, multi-cluster, then infinite clusters

Marginal cluster assignments

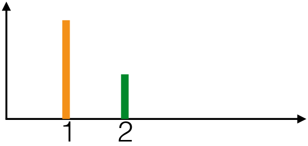
$$\rho_1 \sim \text{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$



Marginal cluster assignments

- Integrate out the frequencies

  $\rho_1 \sim \text{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$

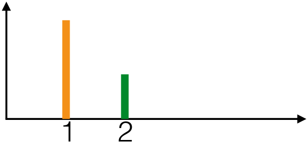


Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \text{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$



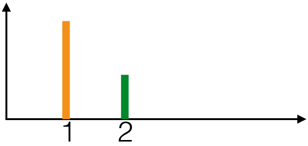
Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \stackrel{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

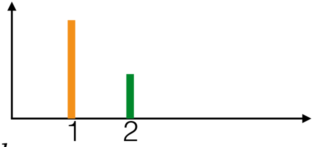
$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$

$$= \int p(z_n = 1, \rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$



Marginal cluster assignments

- Integrate out the frequencies



$$\rho_1 \sim \text{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

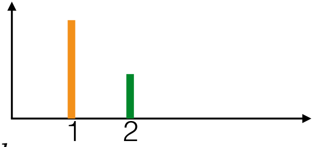
$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$
$$= \int p(z_n = 1 | \rho_1) p(\rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$

Marginal cluster assignments

- Integrate out the frequencies

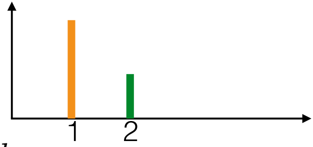
$$\rho_1 \sim \text{Beta}(a_1, a_2),\ z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$
$$= \int p(z_n = 1 | \rho_1) p(\rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$
$$= \int \rho_1 \text{Beta}(\rho_1 | a_{1,n}, a_{2,n}) d\rho_1$$



Marginal cluster assignments

- Integrate out the frequencies



$$\rho_1 \sim \mathrm{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$

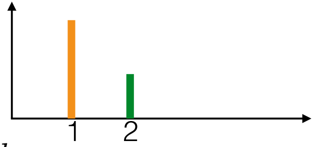
$$= \int p(z_n = 1 | \rho_1) p(\rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$

$$= \int \rho_1 \mathrm{Beta}(\rho_1 | a_{1,n}, a_{2,n}) d\rho_1$$

$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\}, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

Marginal cluster assignments

- Integrate out the frequencies



$$\rho_1 \sim \text{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$

$$= \int p(z_n = 1 | \rho_1) p(\rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$

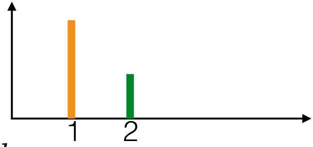
$$= \int \rho_1 \text{Beta}(\rho_1 | a_{1,n}, a_{2,n}) d\rho_1$$

$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\}, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

$$= \int \rho_1 \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \rho_1^{a_{1,n}-1} (1 - \rho_1)^{a_{2,n}-1} d\rho_1$$

Marginal cluster assignments

- Integrate out the frequencies

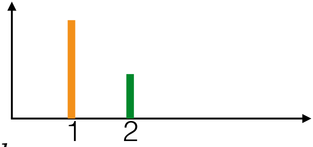


$$\rho_1 \sim \text{Beta}(a_1, a_2),\ z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1})$$
$$= \int p(z_n = 1 | \rho_1) p(\rho_1 | z_1, \ldots, z_{n-1}) d\rho_1$$
$$= \int \rho_1 \text{Beta}(\rho_1 | a_{1,n}, a_{2,n}) d\rho_1$$
$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\ a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$
$$= \int \rho_1 \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \rho_1^{a_{1,n}-1} (1-\rho_1)^{a_{2,n}-1} d\rho_1$$
$$= \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \frac{\Gamma(a_{1,n}+1)\Gamma(a_{2,n})}{\Gamma(a_{1,n} + a_{2,n} + 1)}$$

Marginal cluster assignments

- Integrate out the frequencies



$$\rho_1 \sim \text{Beta}(a_1, a_2),\ z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$\begin{aligned}
p(z_n = 1 | z_1, \ldots, z_{n-1}) \\
&= \int p(z_n = 1 | \rho_1) p(\rho_1 | z_1, \ldots, z_{n-1}) d\rho_1 \\
&= \int \rho_1 \text{Beta}(\rho_1 | a_{1,n}, a_{2,n}) d\rho_1
\end{aligned}$$

$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\ a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

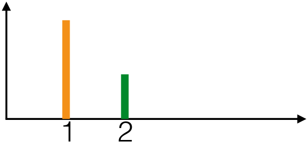
$$\begin{aligned}
&= \int \rho_1 \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \rho_1^{a_{1,n}-1} (1 - \rho_1)^{a_{2,n}-1} d\rho_1 \\
&= \frac{\Gamma(a_{1,n} + a_{2,n})}{\Gamma(a_{1,n})\Gamma(a_{2,n})} \frac{\Gamma(a_{1,n} + 1)\Gamma(a_{2,n})}{\Gamma(a_{1,n} + a_{2,n} + 1)} \\
&= \frac{a_{1,n}}{a_{1,n} + a_{2,n}}
\end{aligned}$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \mathrm{Beta}(a_1, a_2),\ z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



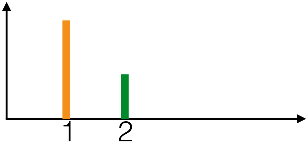
$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\ a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \text{Beta}(a_1, a_2),\, z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

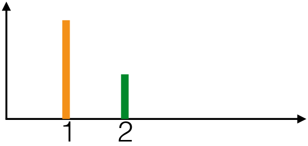
- Pólya urn

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \text{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



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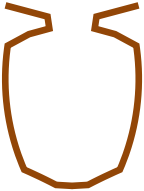
- Pólya urn

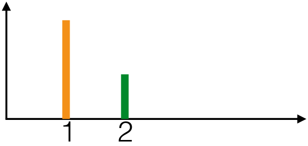


Marginal cluster assignments

- Integrate out the frequencies

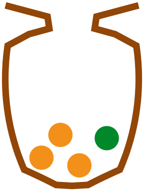
$$\rho_1 \sim \text{Beta}(a_1, a_2),\, z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

- Pólya urn

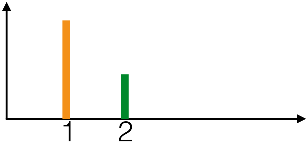


Marginal cluster assignments

- Integrate out the frequencies

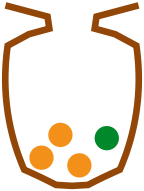
$$\rho_1 \sim \text{Beta}(a_1, a_2),\, z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

- Pólya urn
  - Choose any ball with equal probability

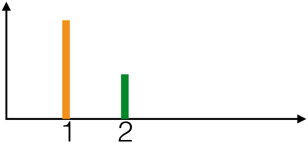


Marginal cluster assignments

- Integrate out the frequencies

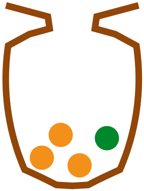
$$\rho_1 \sim \text{Beta}(a_1, a_2),\, z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

- Pólya urn
  - Choose any ball with equal probability
  - Replace and add ball of same color

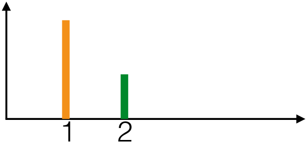


Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \text{Beta}(a_1, a_2), z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\}, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

- Pólya urn
  - Choose any ball with equal probability
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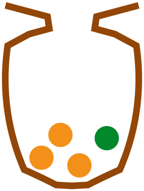


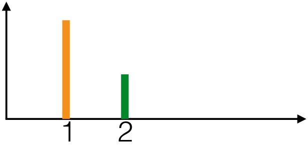
$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}}$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \text{Beta}(a_1, a_2),\ z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\ a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

- Pólya urn
  - Choose any ball with equal probability
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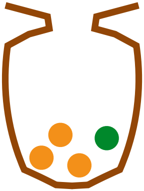


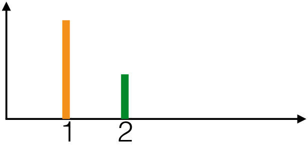
$$\lim_{n\to\infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}}$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_1 \sim \text{Beta}(a_1, a_2),\, z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

- Pólya urn
  - Choose any ball with equal probability
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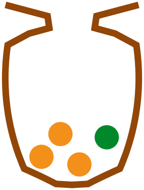


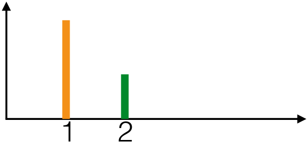
$$\lim_{n \to \infty} \frac{\#\text{ orange}}{\#\text{ total}} = \rho_{\text{orange}} \overset{d}{=} \text{Beta}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

- Integrate out the frequencies

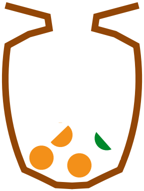
$$\rho_1 \sim \text{Beta}(a_1, a_2),\ z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\ a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

- Pólya urn
  - Choose any ball with equal probability
  - Replace and add ball of same color



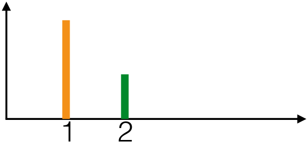
$$\lim_{n \to \infty} \frac{\#\ \text{orange}}{\#\ \text{total}} = \rho_{\text{orange}} \overset{d}{=} \text{Beta}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

- Integrate out the frequencies

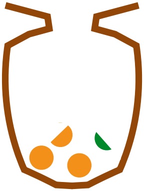
$$\rho_1 \sim \text{Beta}(a_1, a_2),\, z_n \overset{iid}{\sim} \text{Cat}(\rho_1, \rho_2)$$

$$p(z_n = 1 | z_1, \ldots, z_{n-1}) = \frac{a_{1,n}}{a_{1,n} + a_{2,n}}$$



$$a_{1,n} := a_1 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 1\},\, a_{2,n} = a_2 + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = 2\}$$

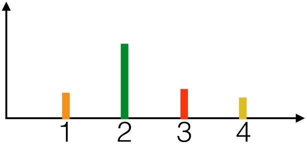
- Pólya urn
  - Choose any ball with prob proportional to its mass
  - Replace and add ball of same color



$$\lim_{n \to \infty} \frac{\#\text{ orange}}{\#\text{ total}} = \rho_{\text{orange}} \overset{d}{=} \text{Beta}(a_{\text{orange}}, a_{\text{green}})$$

Marginal cluster assignments

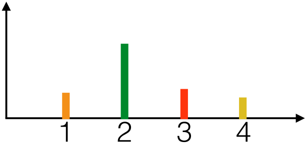
- Integrate out the frequencies



Marginal cluster assignments

- Integrate out the frequencies

$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}),\, z_n \overset{iid}{\sim} \text{Cat}(\rho_{1:K})$

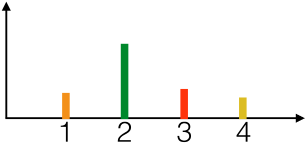


Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}),\ z_n \overset{iid}{\sim} \text{Cat}(\rho_{1:K})$$

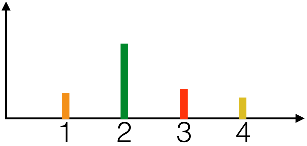
$$p(z_n = k | z_1, \ldots, z_{n-1}) = \frac{a_{k,n}}{\sum_{j=1}^{K} a_{j,n}}$$



Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}),\ z_n \overset{iid}{\sim} \text{Cat}(\rho_{1:K})$$



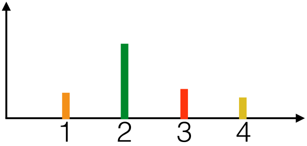
$$p(z_n = k | z_1, \ldots, z_{n-1}) = \frac{a_{k,n}}{\sum_{j=1}^{K} a_{j,n}}$$

$$a_{k,n} := a_k + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = k\}$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}),\, z_n \overset{iid}{\sim} \text{Cat}(\rho_{1:K})$$



$$p(z_n = k | z_1, \ldots, z_{n-1}) = \frac{a_{k,n}}{\sum_{j=1}^{K} a_{j,n}}$$

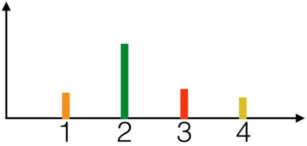
$$a_{k,n} := a_k + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = k\}$$

- multivariate Pólya urn

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}),\ z_n \overset{iid}{\sim} \text{Cat}(\rho_{1:K})$$



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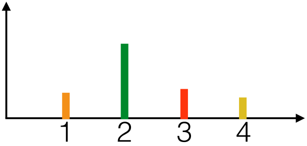
- multivariate Pólya urn

Marginal cluster assignments

- Integrate out the frequencies



$$\rho_{1:K} \sim \mathrm{Dirichlet}(a_{1:K}), \; z_n \overset{iid}{\sim} \mathrm{Cat}(\rho_{1:K})$$

$$p(z_n = k | z_1, \ldots, z_{n-1}) = \frac{a_{k,n}}{\sum_{j=1}^{K} a_{j,n}}$$

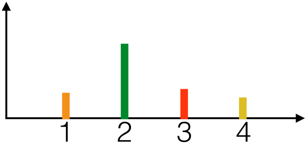
$$a_{k,n} := a_k + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = k\}$$

- multivariate Pólya urn
  - Choose any ball with prob proportional to its mass

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}),\ z_n \overset{iid}{\sim} \text{Cat}(\rho_{1:K})$$



$$p(z_n = k | z_1, \ldots, z_{n-1}) = \frac{a_{k,n}}{\sum_{j=1}^{K} a_{j,n}}$$

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- multivariate Pólya urn
  - Choose any ball with prob proportional to its mass
  - Replace and add ball of same color

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}),\ z_n \overset{iid}{\sim} \text{Cat}(\rho_{1:K})$$

$$p(z_n = k | z_1, \ldots, z_{n-1}) = \frac{a_{k,n}}{\sum_{j=1}^K a_{j,n}}$$

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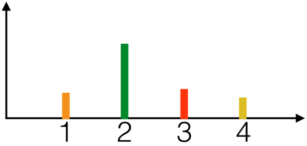


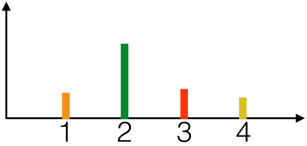
- multivariate Pólya urn
  - Choose any ball with prob proportional to its mass
  - Replace and add ball of same color

$$\lim_{n \to \infty} \frac{(\#\ \text{orange},\ \#\ \text{green},\ \#\ \text{red},\ \#\ \text{yellow})}{\#\ \text{total}}$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}),\ z_n \overset{iid}{\sim} \text{Cat}(\rho_{1:K})$$



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- multivariate Pólya urn
  - Choose any ball with prob proportional to its mass
  - Replace and add ball of same color

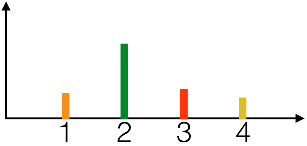
$$\lim_{n \to \infty} \frac{(\#\ \text{orange},\ \#\ \text{green},\ \#\ \text{red},\ \#\ \text{yellow})}{\#\ \text{total}}$$

$$\to (\rho_{\text{orange}}, \rho_{\text{green}}, \rho_{\text{red}}, \rho_{\text{yellow}})$$

Marginal cluster assignments

- Integrate out the frequencies

$$\rho_{1:K} \sim \text{Dirichlet}(a_{1:K}),\, z_n \overset{iid}{\sim} \text{Cat}(\rho_{1:K})$$



$$p(z_n = k | z_1, \ldots, z_{n-1}) = \frac{a_{k,n}}{\sum_{j=1}^{K} a_{j,n}}$$

$$a_{k,n} := a_k + \sum_{m=1}^{n-1} \mathbf{1}\{z_m = k\}$$

- multivariate Pólya urn
  - Choose any ball with prob proportional to its mass
  - Replace and add ball of same color

$$\lim_{n \to \infty} \frac{(\#\text{ orange, } \#\text{ green, } \#\text{ red, } \#\text{ yellow})}{\#\text{ total}}$$

$$\to (\rho_{\text{orange}}, \rho_{\text{green}}, \rho_{\text{red}}, \rho_{\text{yellow}})$$

$$\overset{d}{=} \text{Dirichlet}(a_{\text{orange}}, a_{\text{green}}, a_{\text{red}}, a_{\text{yellow}})$$

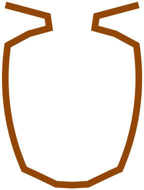
Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

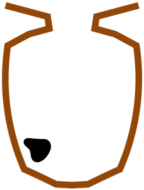
- Hoppe urn / Blackwell-MacQueen urn



[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

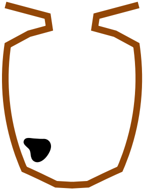
- Hoppe urn / Blackwell-MacQueen urn



[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn

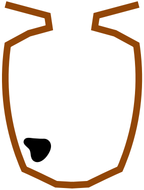


  - Choose ball with prob proportional to its mass

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn

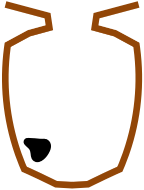


- Choose ball with prob proportional to its mass
  - If black, replace and add ball of new color

[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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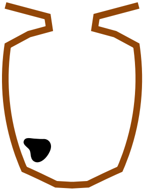
[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



- Choose ball with prob proportional to its mass
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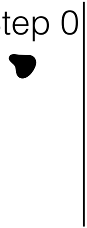
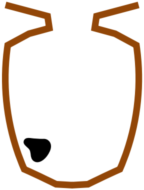
Step 0



[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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  - If black, replace and add ball of new color
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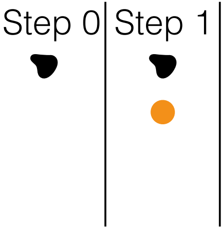
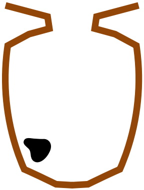


[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



- Choose ball with prob proportional to its mass
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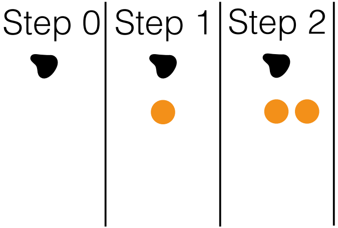
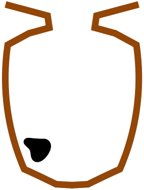


[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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  - If black, replace and add ball of new color
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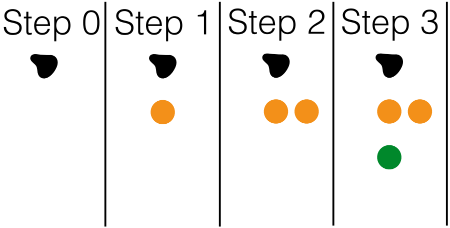
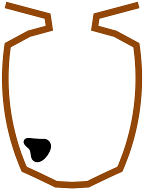


[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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  - If black, replace and add ball of new color
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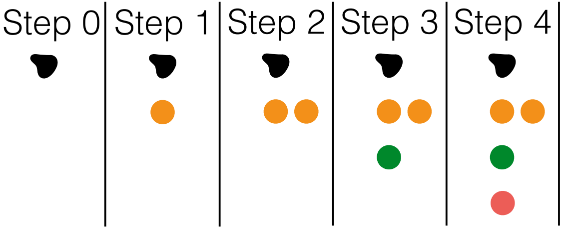
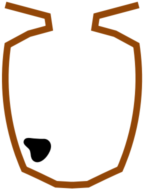


[Blackwell, MacQueen 1973; Hoppe 1984]

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



- Choose ball with prob proportional to its mass
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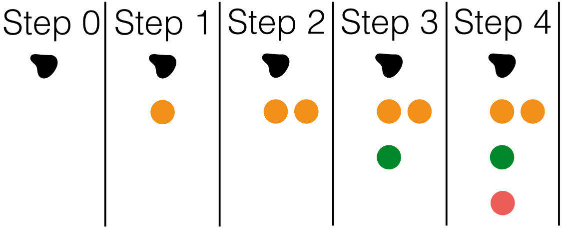
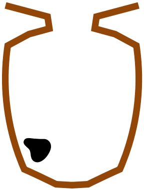


$$(\#\mathrm{orange}, \#\mathrm{other}) = \mathrm{PolyaUrn}(1, \alpha)$$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



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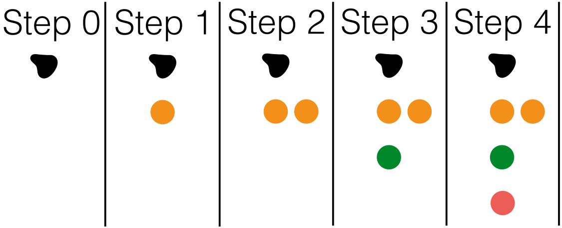
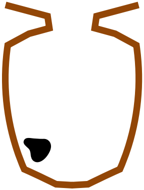


$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

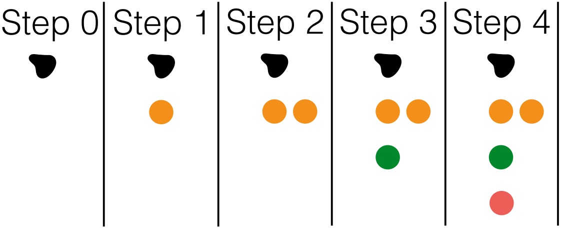
- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



- Choose ball with prob proportional to its mass
  - If black, replace and add ball of new color
  - Else, replace and add ball of same color

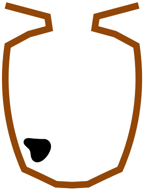


$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$

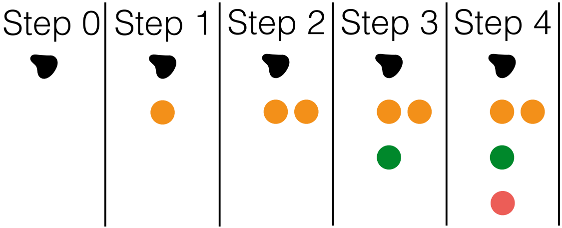
- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$
- not orange, green: $(\#\text{red}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



- Choose ball with prob proportional to its mass
  - If black, replace and add ball of new color
  - Else, replace and add ball of same color



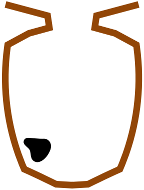
$$V_k \stackrel{iid}{\sim} \text{Beta}(1, \alpha)$$

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$
- not orange, green: $(\#\text{red}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



- Choose ball with prob proportional to its mass
  - If black, replace and add ball of new color
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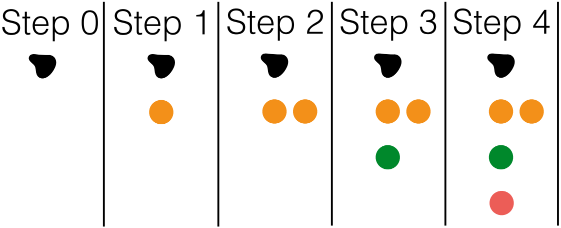


$$V_k \stackrel{iid}{\sim} \text{Beta}(1, \alpha)$$

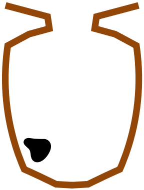
$$\rho_1 = V_1$$

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$
- not orange, green: $(\#\text{red}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



- Choose ball with prob proportional to its mass
  - If black, replace and add ball of new color
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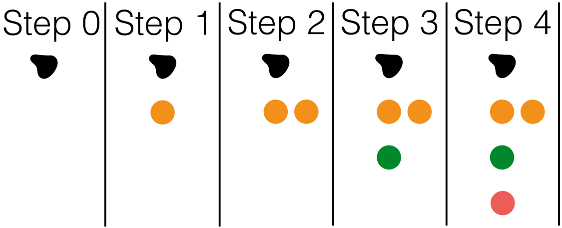


$$V_k \overset{iid}{\sim} \text{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$

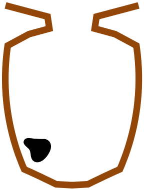
$$\rho_2 = (1 - V_1)V_2$$

$$(\#\text{orange}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$$

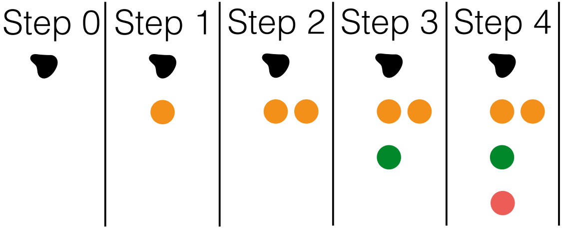
- not orange: $(\#\text{green}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$
- not orange, green: $(\#\text{red}, \#\text{other}) = \text{PolyaUrn}(1, \alpha)$

Marginal cluster assignments

- Hoppe urn / Blackwell-MacQueen urn



  - Choose ball with prob proportional to its mass
    - If black, replace and add ball of new color
    - Else, replace and add ball of same color



$$V_k \stackrel{iid}{\sim} \mathrm{Beta}(1, \alpha)$$

$$\rho_1 = V_1$$
$$\rho_2 = (1 - V_1)V_2$$
$$\rho_3 = [\prod_{k=1}^{2} (1 - V_k)]V_3$$

$$(\#\mathrm{orange}, \#\mathrm{other}) = \mathrm{PolyaUrn}(1, \alpha)$$

- not orange: $(\#\mathrm{green}, \#\mathrm{other}) = \mathrm{PolyaUrn}(1, \alpha)$
- not orange, green: $(\#\mathrm{red}, \#\mathrm{other}) = \mathrm{PolyaUrn}(1, \alpha)$

Chinese restaurant process

Chinese restaurant process

- Same thing we just did

Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant

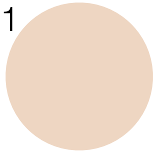
Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there

Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

Chinese restaurant process



- Same thing we just did
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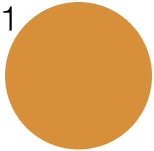
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

Chinese restaurant process



- Same thing we just did
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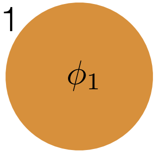
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

Chinese restaurant process



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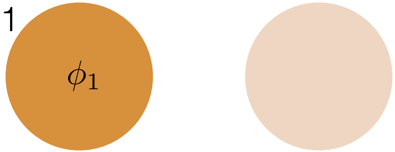
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

Chinese restaurant process



- Same thing we just did
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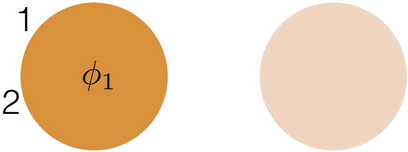
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

Chinese restaurant process



- Same thing we just did
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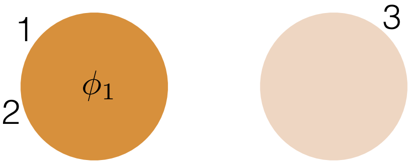
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

Chinese restaurant process



- Same thing we just did
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  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

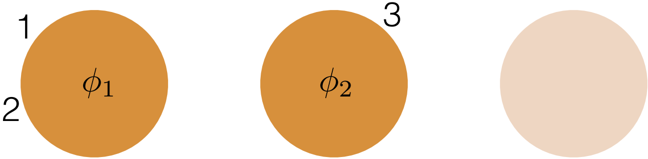
Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

Chinese restaurant process

- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

Chinese restaurant process



- Same thing we just did
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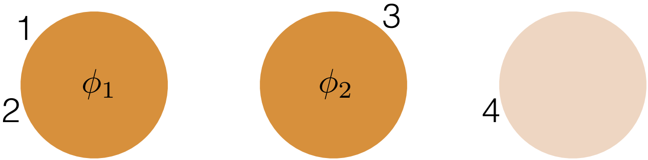
Chinese restaurant process



- Same thing we just did
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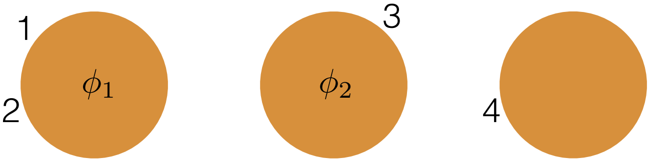
  - Forms new table with prob proportional to α

Chinese restaurant process



- Same thing we just did
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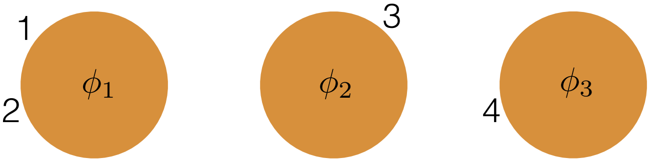
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

Chinese restaurant process



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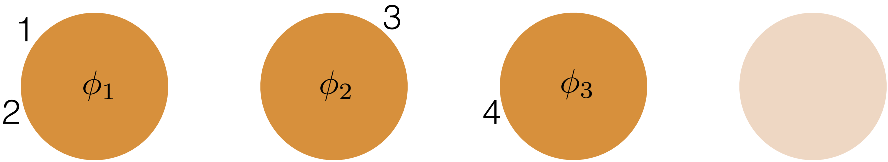
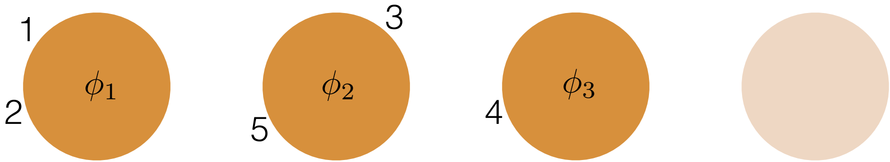
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

Chinese restaurant process



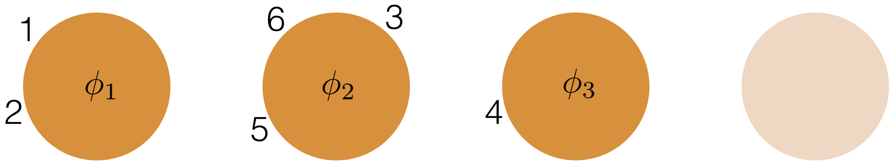
- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

Chinese restaurant process



- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

Chinese restaurant process



- Same thing we just did
- Each customer walks into the restaurant
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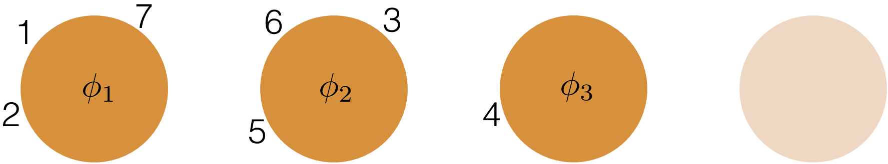
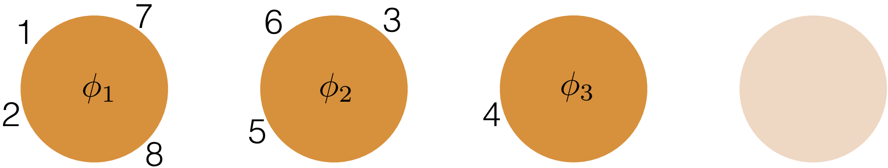
  - Forms new table with prob proportional to α

Chinese restaurant process



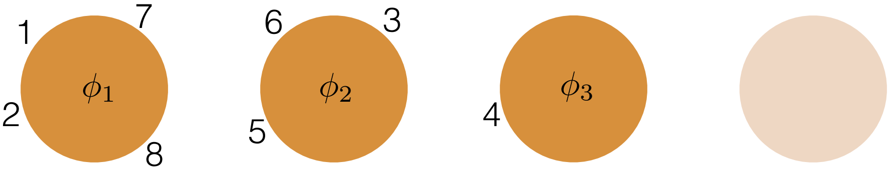
- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

Chinese restaurant process



- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

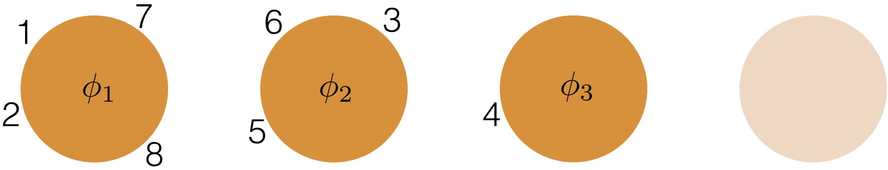
Chinese restaurant process



- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α

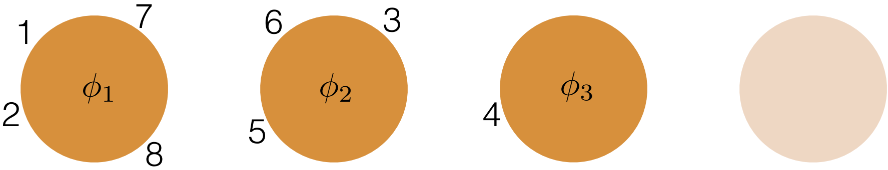
[Aldous 1983]

Chinese restaurant process



- Same thing we just did
- Each customer walks into the restaurant
    - Sits at existing table with prob proportional to # people there
    - Forms new table with prob proportional to α
- Marginal for the Categorical likelihood with GEM prior

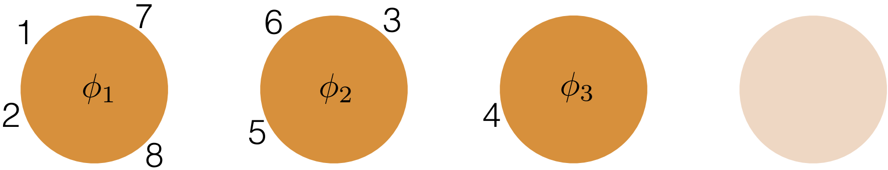
Chinese restaurant process



- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α
- Marginal for the Categorical likelihood with GEM prior

$$z_1 = z_2 = z_7 = z_8 = 1, z_3 = z_5 = z_6 = 2, z_4 = 3$$

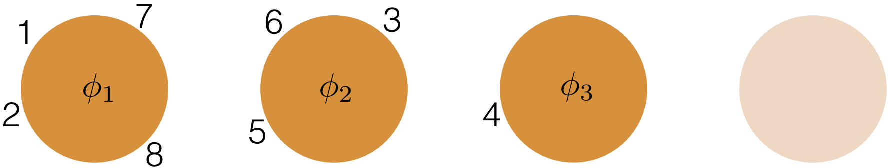
Chinese restaurant process



- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α
- Marginal for the Categorical likelihood with GEM prior

$$z_1 = z_2 = z_7 = z_8 = 1, z_3 = z_5 = z_6 = 2, z_4 = 3$$
$$\Rightarrow \Pi_8 = \{\{1, 2, 7, 8\}, \{3, 5, 6\}, \{4\}\}$$

Chinese restaurant process

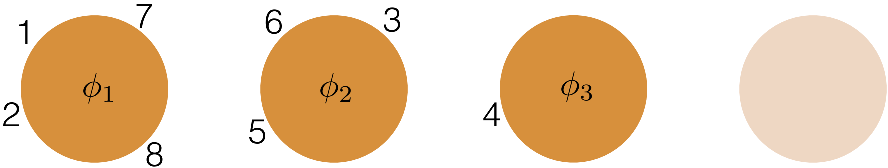


- Same thing we just did
- Each customer walks into the restaurant
  - Sits at existing table with prob proportional to # people there
  - Forms new table with prob proportional to α
- Marginal for the Categorical likelihood with GEM prior

$z_1 = z_2 = z_7 = z_8 = 1, z_3 = z_5 = z_6 = 2, z_4 = 3$
$\Rightarrow \Pi_8 = \{\{1,2,7,8\},\{3,5,6\},\{4\}\}$

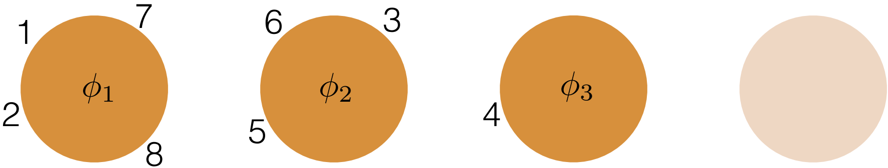
- *Partition of [8]*: set of mutually exclusive & exhaustive sets of $[8] = \{1, \ldots, 8\}$

Chinese restaurant process



- Probability of this seating:

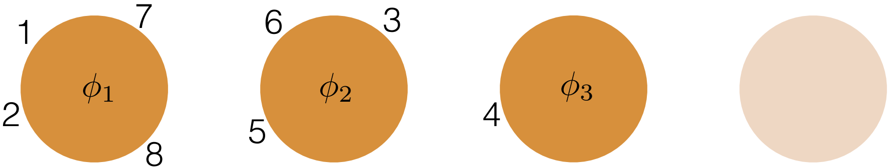
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha}$$

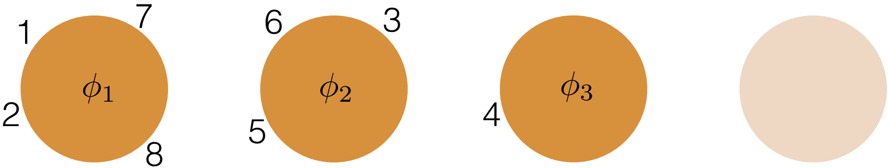
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha + 1}$$

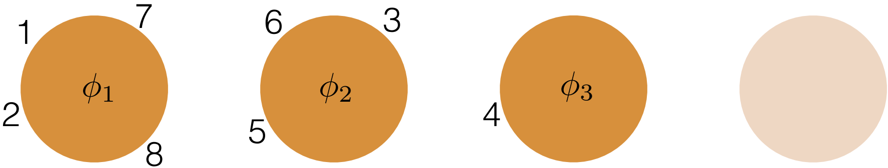
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha + 1} \cdot \frac{\alpha}{\alpha + 2}$$

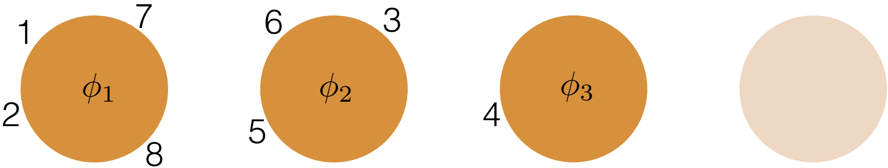
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3}$$

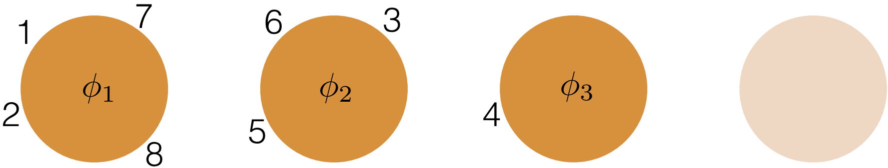
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4}$$

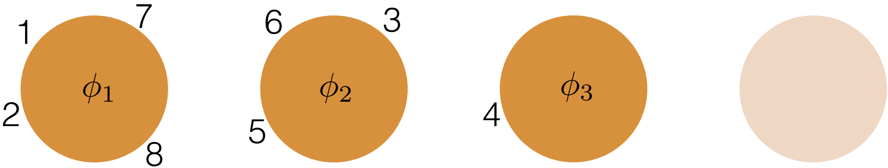
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5}$$

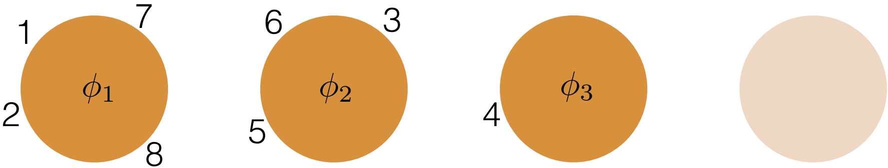
Chinese restaurant process



- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6}$$

Chinese restaurant process



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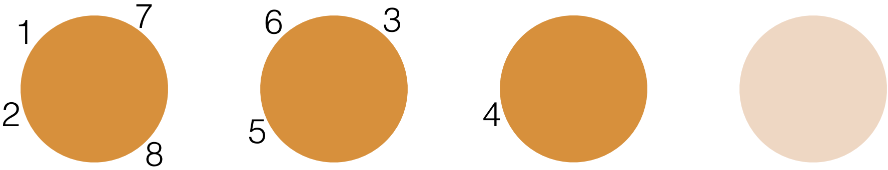
$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$

Chinese restaurant process

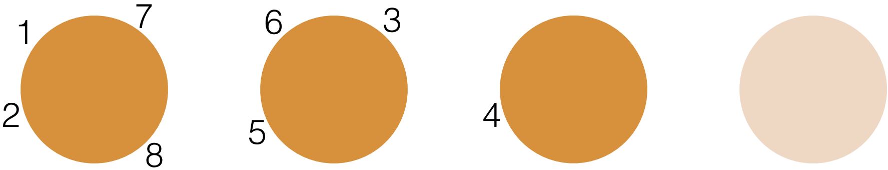


- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$

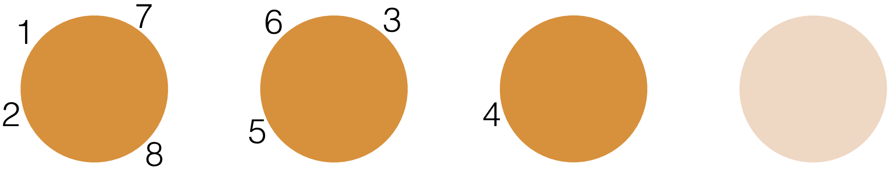
- Probability of *N* customers (K_N tables, n_k at table *k*):

Chinese restaurant process



- Probability of this seating:
$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$
- Probability of *N* customers (K_N tables, n_k at table *k*):

Chinese restaurant process



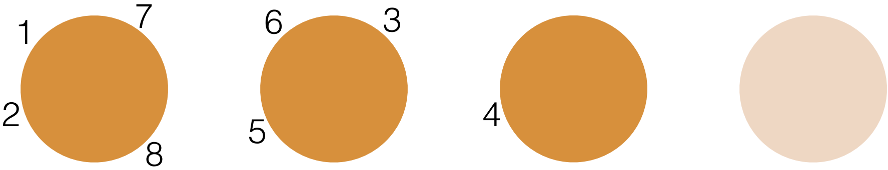
- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$

- Probability of N customers (K_N tables, n_k at table k):

$$\frac{\quad}{\alpha \cdots (\alpha + N - 1)}$$

Chinese restaurant process



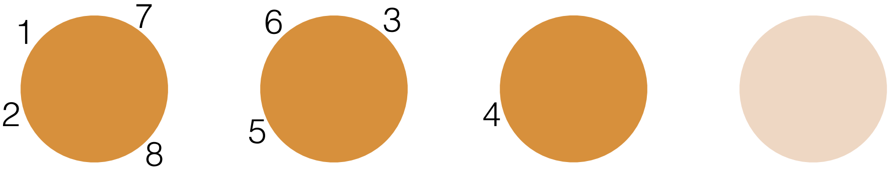
- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$

- Probability of N customers (K_N tables, n_k at table k):

$$\frac{\alpha^{K_N}}{\alpha \cdots (\alpha + N - 1)}$$

Chinese restaurant process



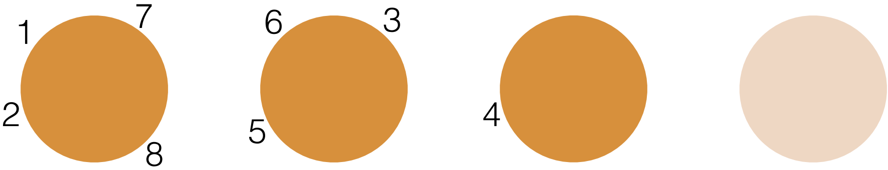
- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$

- Probability of N customers (K_N tables, n_k at table k):

$$\frac{\alpha^{K_N}}{\alpha \cdots (\alpha + N - 1)}$$

Chinese restaurant process



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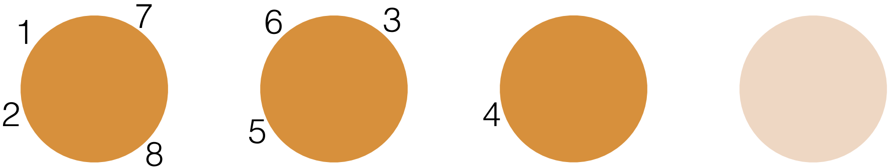
- Probability of N customers (K_N tables, n_k at table k):

$$\frac{\alpha^{K_N}}{\alpha \cdots (\alpha + N - 1)}$$

Chinese restaurant process



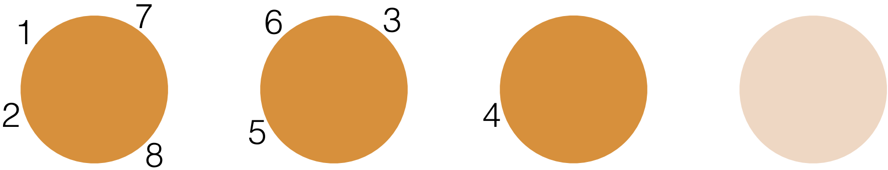
- Probability of this seating:

$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$

- Probability of *N* customers (K_N tables, n_k at table *k*):

$$\frac{\alpha^{K_N}}{\alpha \cdots (\alpha + N - 1)}$$

Chinese restaurant process



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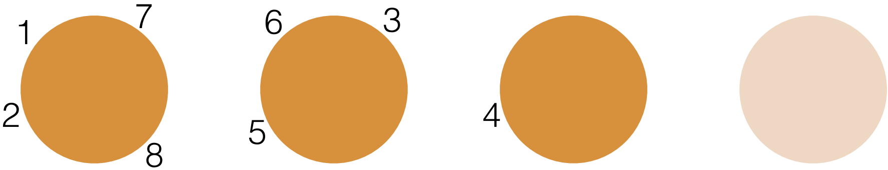
- Probability of *N* customers (K_N tables, n_k at table *k*):

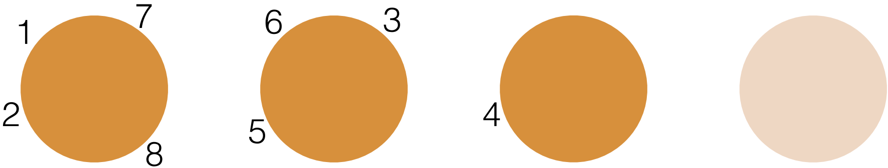
$$\frac{\alpha^{K_N}}{\alpha \cdots (\alpha + N - 1)}$$

Chinese restaurant process



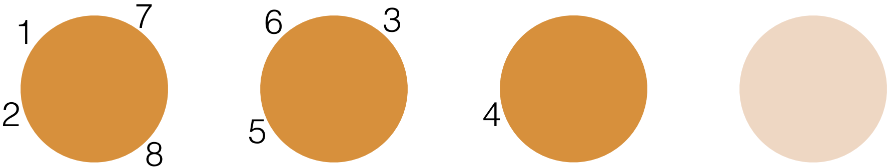
- Probability of this seating:
$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$
- Probability of *N* customers (K_N tables, n_k at table *k*):
$$\frac{\alpha^{K_N} \prod_{k=1}^{K_N} (n_k - 1)!}{\alpha \cdots (\alpha + N - 1)}$$

Chinese restaurant process



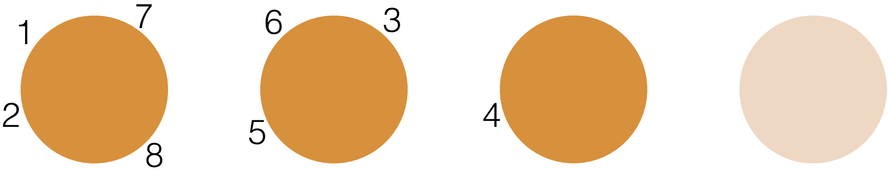
- Probability of this seating:
  $$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$
- Probability of N customers (K_N tables, $\#C$ at table C):
  $$\frac{\alpha^{K_N} \prod_{C \in \Pi_N} (\#C - 1)!}{\alpha \cdots (\alpha + N - 1)}$$

Chinese restaurant process



- Probability of this seating:
  $$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$
- Probability of N customers (K_N tables, $\#C$ at table C):
  $$\frac{\alpha^{K_N} \prod_{C \in \Pi_N} (\#C - 1)!}{\alpha \cdots (\alpha + N - 1)} = \mathbb{P}(\Pi_N = \pi_N)$$

Chinese restaurant process



- Probability of this seating:

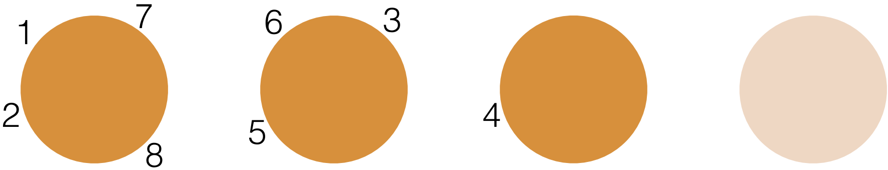
$$\frac{\alpha}{\alpha} \cdot \frac{1}{\alpha+1} \cdot \frac{\alpha}{\alpha+2} \cdot \frac{\alpha}{\alpha+3} \cdot \frac{1}{\alpha+4} \cdot \frac{2}{\alpha+5} \cdot \frac{2}{\alpha+6} \cdot \frac{3}{\alpha+7}$$

- Probability of N customers (K_N tables, $\#C$ at table C):

$$\frac{\alpha^{K_N} \prod_{C \in \Pi_N} (\#C - 1)!}{\alpha \cdots (\alpha + N - 1)} = \mathbb{P}(\Pi_N = \pi_N)$$

- Prob doesn't depend on customer order: *exchangeable*

Chinese restaurant process



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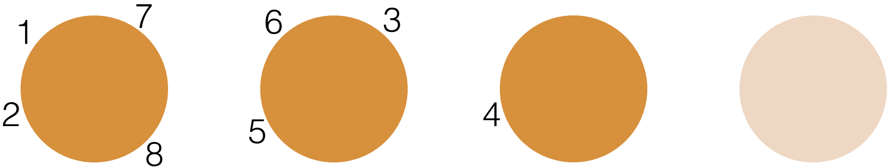
$$\mathbb{P}(\Pi_8 = \{\{1,2,7,8\},\{3,5,6\},\{4\}\}) = \mathbb{P}(\Pi_8 = \{\{2,3,8,1\},\{4,6,7\},\{5\}\})$$

Chinese restaurant process



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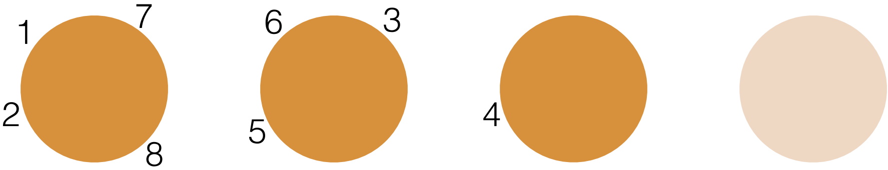
Chinese restaurant process



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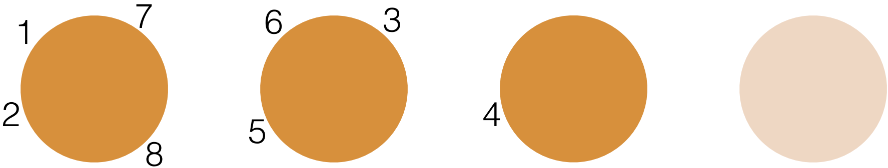
- Can always pretend *n* is the last customer and calculate $p(\Pi_N | \Pi_{N,\text{-}n})$
    - e.g. $\Pi_{8,\text{-}5} = \{\{1,2,7,8\},\{3,6\},\{4\}\}$

Chinese restaurant process



- Probability of N customers (K_N tables, $\#C$ at table C):

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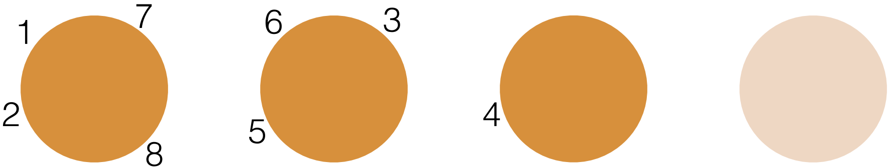
- So:

$$p(\Pi_N | \Pi_{N,-n}) =$$

Chinese restaurant process



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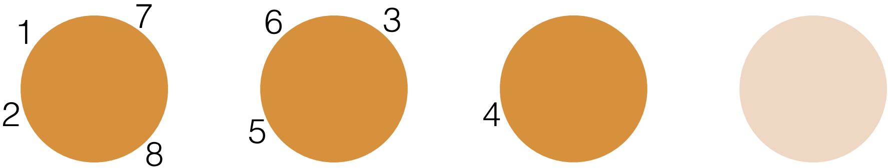
Chinese restaurant process



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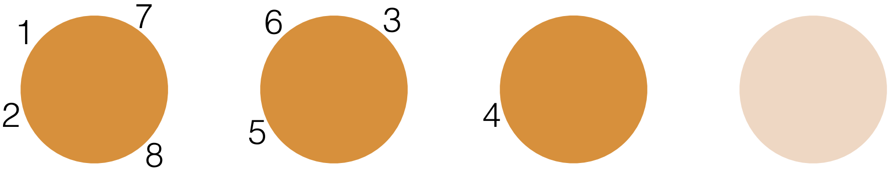
Chinese restaurant process



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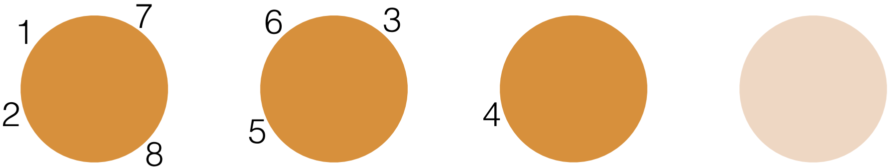
$$p(\Pi_N | \Pi_{N,-n}) = \begin{cases} \frac{\#C}{\alpha + N - 1} & \text{if } n \text{ joins cluster } C \\ & \text{if } n \text{ starts a new cluster} \end{cases}$$

Chinese restaurant process



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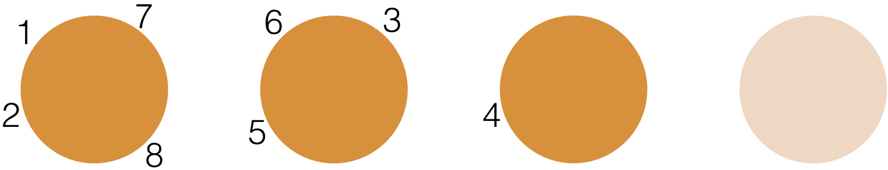

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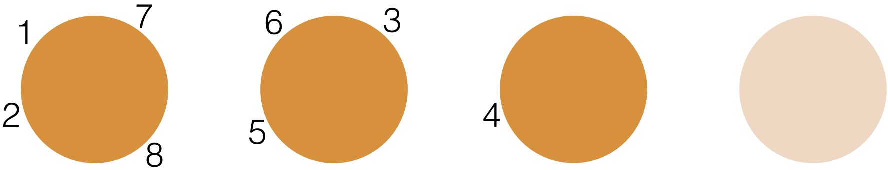
- Gibbs sampling review:

Chinese restaurant process



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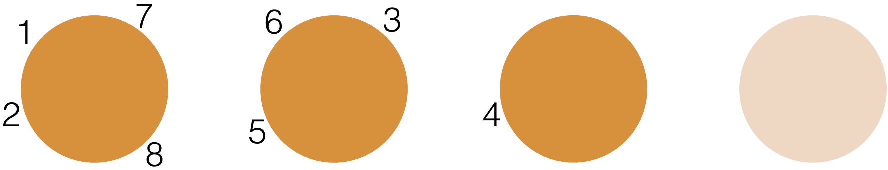
- Gibbs sampling review: target distribution $p(v_1, v_2, v_3)$

Chinese restaurant process



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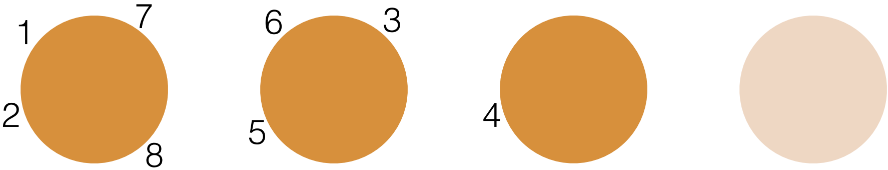
  - Start: $v_1^{(0)}, v_2^{(0)}, v_3^{(0)}$

Chinese restaurant process



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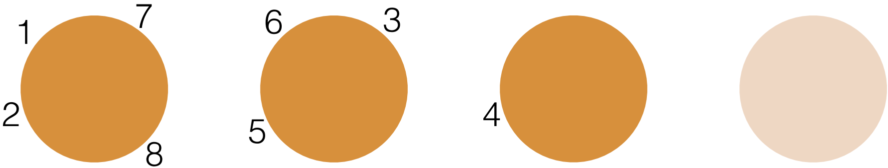
  - t th step: $v_1^{(t)} \sim p(v_1 | v_2^{(t-1)}, v_3^{(t-1)})$

Chinese restaurant process



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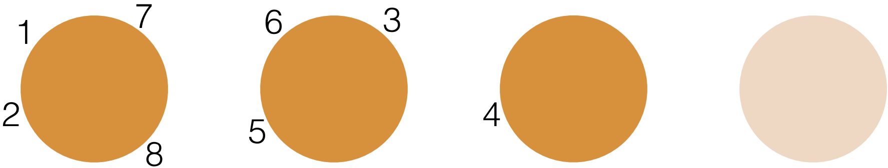
$$v_2^{(t)} \sim p(v_2 | v_1^{(t)}, v_3^{(t-1)})$$

Chinese restaurant process



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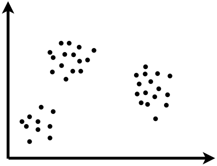
CRP mixture model: inference

CRP mixture model: inference

- Data $x_{1:N}$



CRP mixture model: inference

- Data $x_{1:N}$

CRP mixture model: inference

- Data $x_{1:N}$
- Generative model

CRP mixture model: inference

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- Generative model
  $\Pi_N \sim \mathrm{CRP}(N, \alpha)$

CRP mixture model: inference

- Data $x_{1:N}$

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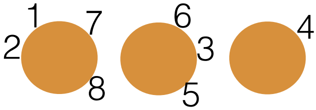

CRP mixture model: inference

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- Generative model

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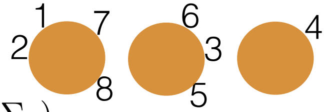
  $\forall C \in \Pi_N, \mu_C \stackrel{iid}{\sim} \mathcal{N}(\mu_0, \Sigma_0)$



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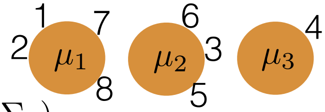

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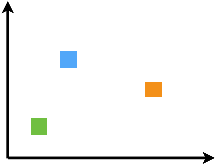

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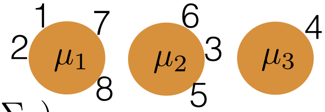

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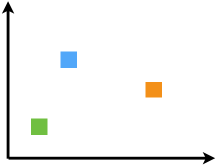

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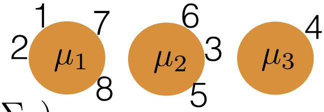

CRP mixture model: inference

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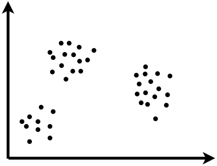

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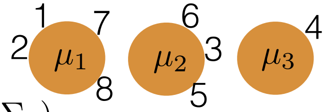

CRP mixture model: inference

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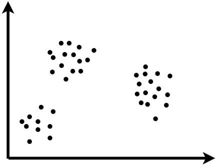

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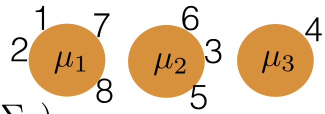
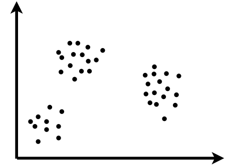


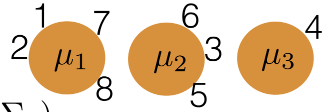
- Want: posterior

CRP mixture model: inference

- Data $x_{1:N}$



- Generative model



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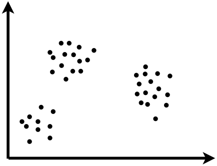
- Want: posterior $p(\Pi_N | x_{1:N})$

CRP mixture model: inference

- Data $x_{1:N}$



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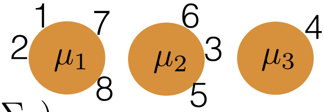
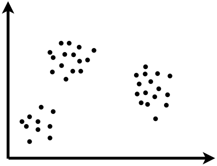


- Want: posterior $p(\Pi_N | x_{1:N})$
- Gibbs sampler:

CRP mixture model: inference

- Data $x_{1:N}$



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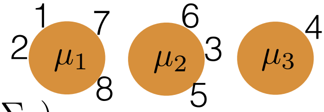


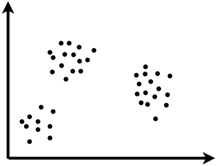
- Want: posterior $p(\Pi_N | x_{1:N})$

- Gibbs sampler:

$$p(\Pi_N | \Pi_{N,\text{-}n}, x)$$

CRP mixture model: inference

- Data $x_{1:N}$



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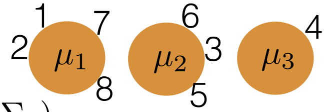

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CRP mixture model: inference

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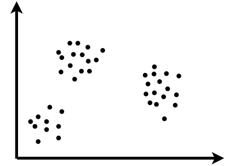

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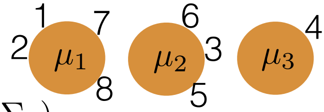

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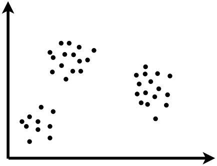

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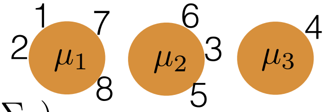

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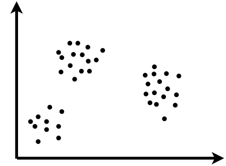

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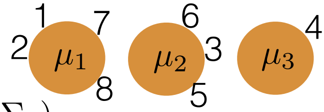

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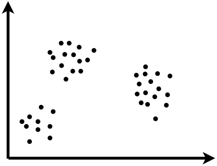

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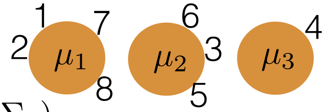

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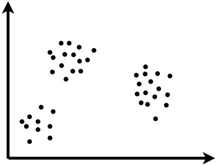

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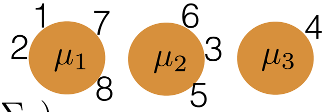

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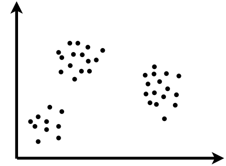

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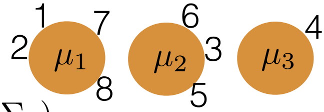

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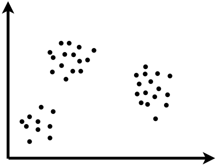

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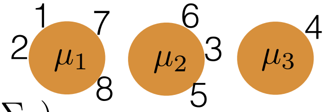

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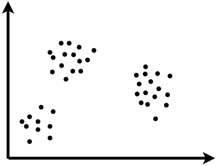

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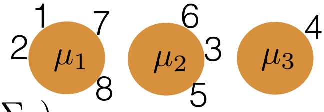

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[MacEachern 1994; Neal 1992; Neal 2000]

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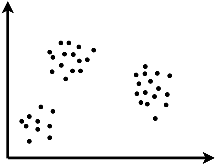

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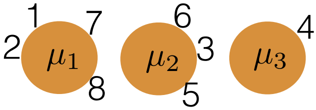

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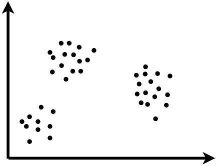

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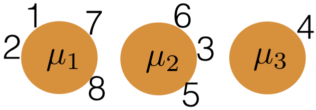

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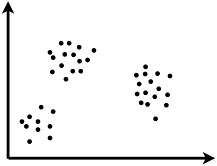

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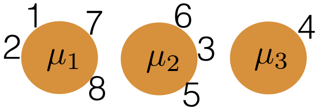

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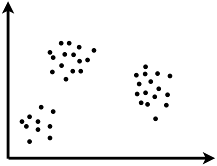

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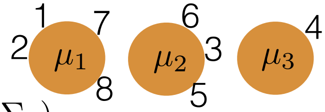

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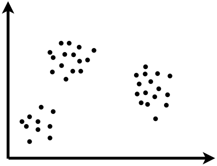

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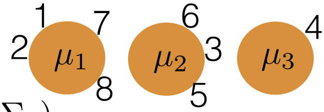

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[demo]

[MacEachern 1994; Neal 1992; Neal 2000]

Conclusions

- Reviewed Gaussian Mixture Modeling
- GEM distribution is an infinite extension of the Dirichlet
- DPMM is a generative process using the GEM on cluster priors
- Stick-Breaking is a representation of the GEM or Dirichlet prior
- Poyla Urn is a representation of categorical marginals with Beta or Dirichlet prior
- Hoppe-Urn is a finite representation of the marginal with GEM prior
- CRP is a finite representation of the marginal with GEM prior

Thanks to borrowed slides from Tamara Broderick

Summary

- Reviewed Gaussian Mixture Modeling
- GEM distribution is an infinite extension of the Dirichlet
- DPMM is a generative process using the GEM on cluster priors
- Stick-Breaking is a representation of the GEM or Dirichlet prior
- (mulitvariate) Poyla Urn is a representation of categorical marginals with Beta (or Dirichlet) prior
- Hoppe-Urn is a finite representation of the marginal with GEM prior
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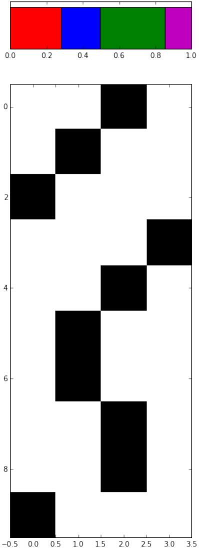
Motivating Example

Many images each with some subset of 4 objects

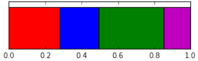
Outline

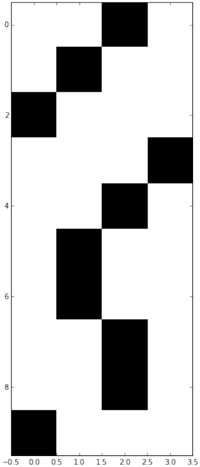
From Clustering to Latent Feature Allocation



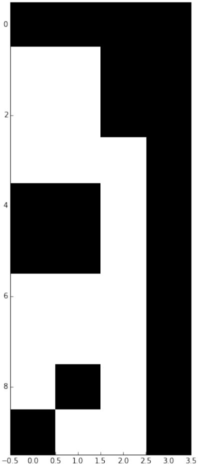
- Write cluster assignments as a binary matrix: $Z_{i,k} = 1$ if sample i belongs to cluster k

From Clustering to Latent Feature Allocation





- Write cluster assignments as a binary matrix: $Z_{i,k} = 1$ if sample i belongs to cluster k
- what if samples could belong to multiple latent groups?



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$$\pi_k|\alpha \sim \frac{\alpha}{K}, 1 \qquad (1)$$

$$z_{i,k}|\pi_k \sim \pi_k \qquad (2)$$

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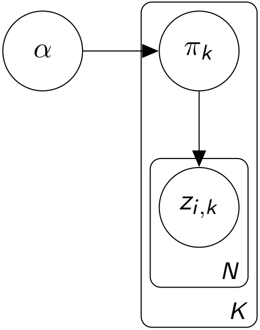

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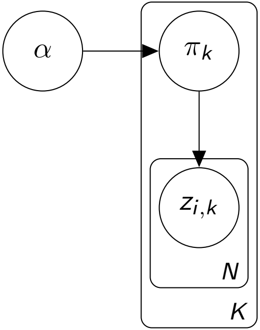

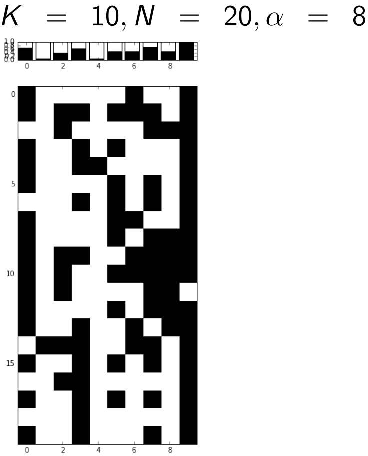

Marginal on Z

for finite K

Model:

$$\pi_k | \alpha \sim \frac{\alpha}{K}, 1$$

$$z_{i,k} | \pi_k \sim \pi_k$$

Marginal on Z

for finite K

Model:

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Recall:

$$\mathrm{B}(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$
$$\Gamma(m) = (m-1)! m \in \mathcal{Z}$$
$$\Gamma(x) = x\Gamma(x-1) x > 0$$

Marginal on Z

for finite K

Model:

$$\pi_k|\alpha \sim \frac{\alpha}{K}, 1$$

$$z_{i,k}|\pi_k \sim \pi_k$$

Recall:

$$\mathrm{B}(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$\Gamma(m) = (m-1)! m \in \mathcal{Z}$$

$$\Gamma(x) = x\Gamma(x-1) x > 0$$

So:

$$P(Z) = \prod_{k=1}^{K} \int \left(\prod_{i=1}^{N} p(z_{i,k}|\pi_k) \right) p(\pi_k) d\pi_k$$

$$= \prod_{k=1}^{K} \frac{\mathrm{B}(n_k + \frac{\alpha}{K}, N - n_k + 1)}{\mathrm{B}(\frac{\alpha}{K}, 1)}$$

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Marginal on Z

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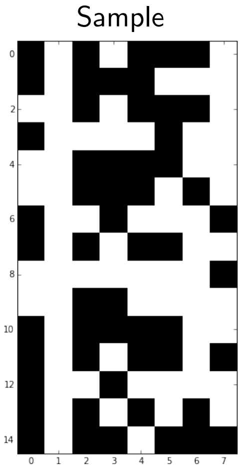
- Follows from Beta-Binomial Conjugacy
- Exchangeable, depends only on $n_k = \sum_{i=1}^{N} z_{i,k}$

Left Ordered Form



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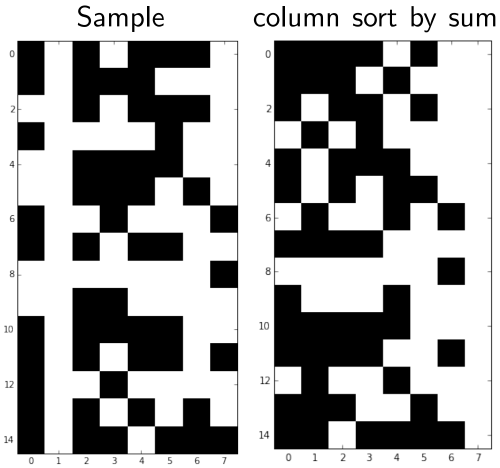

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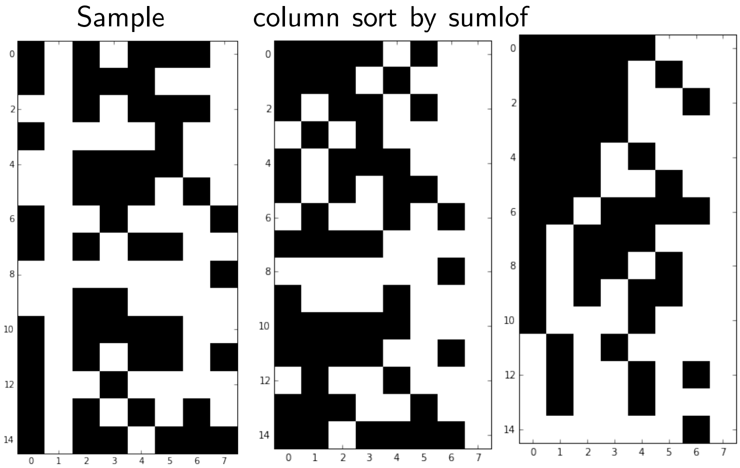

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- Also, $\frac{\alpha}{i}$ new features

Indian Buffet Process

sampling scheme for marginal of $z_{i,k}|\alpha$

First Customer: Sample $\frac{\alpha}{i}$ dishes

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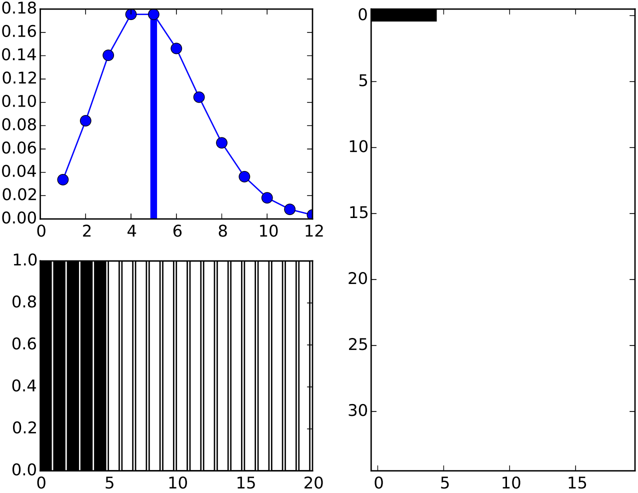
Properties:

- Effective dimension, $K_+ \sim \alpha \sum_{i=1}^{N} \frac{1}{i}$
- Number of dishes sampled by each customer is α by exchangeability

IBP Sampling

$\alpha = 5$



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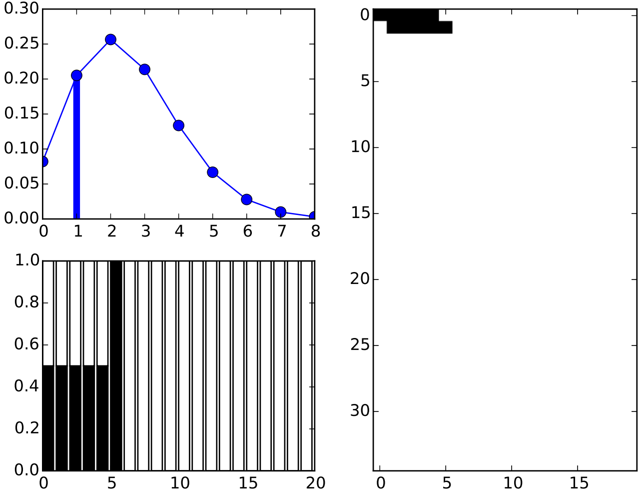

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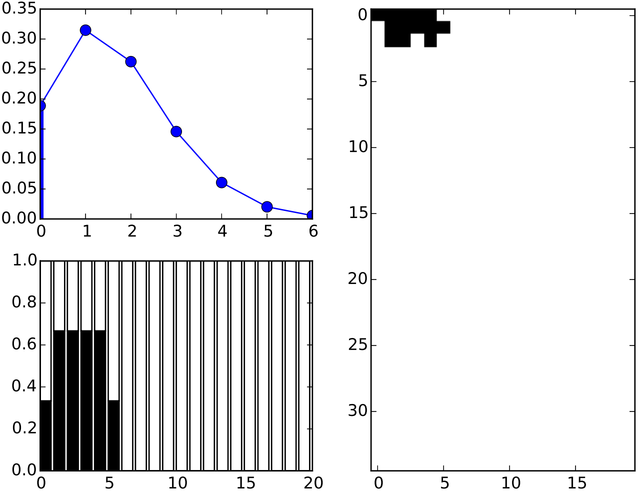

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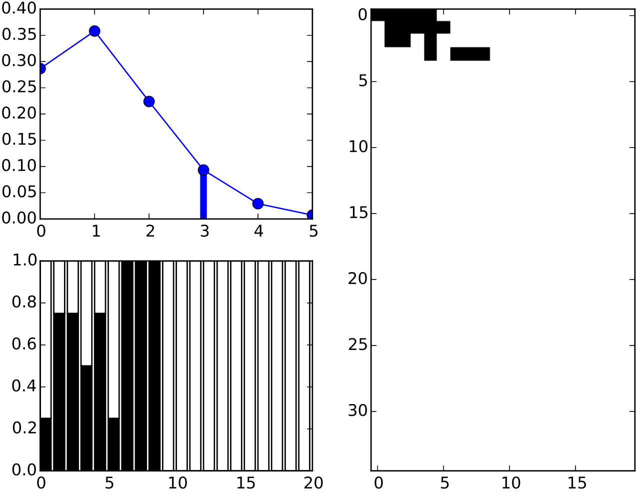

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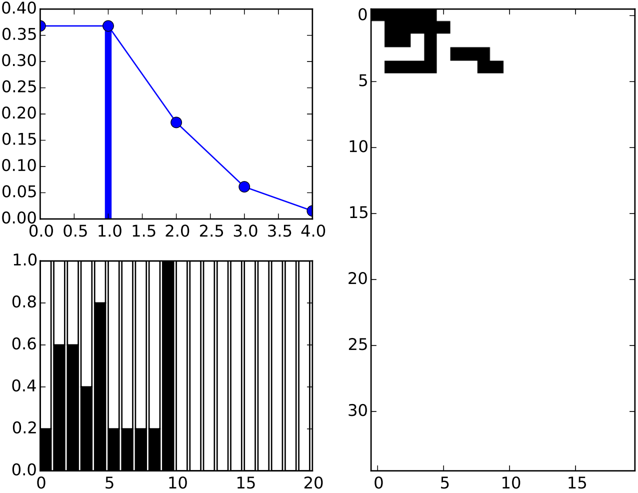

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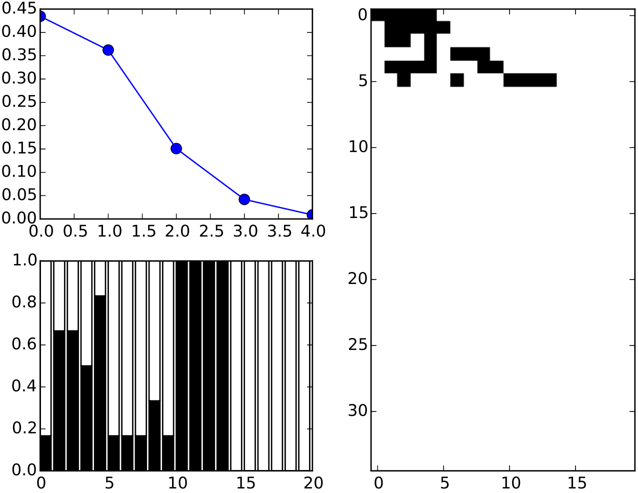

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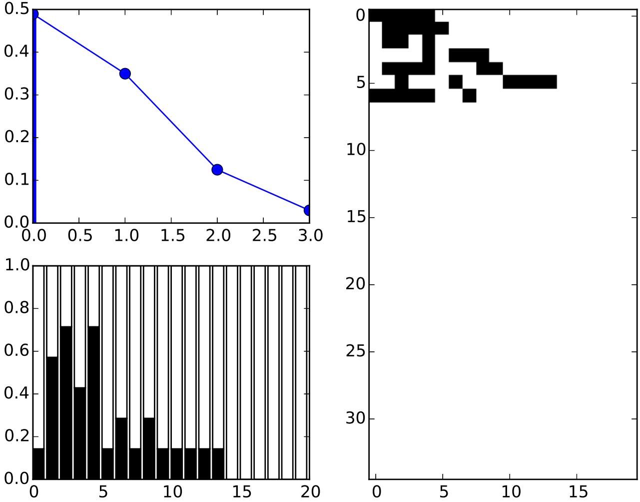

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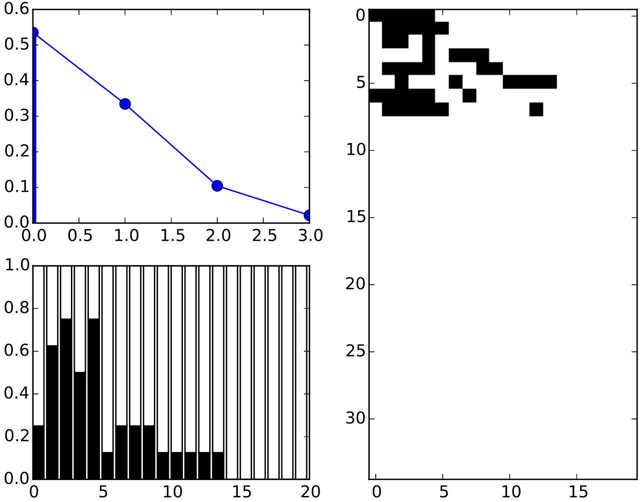

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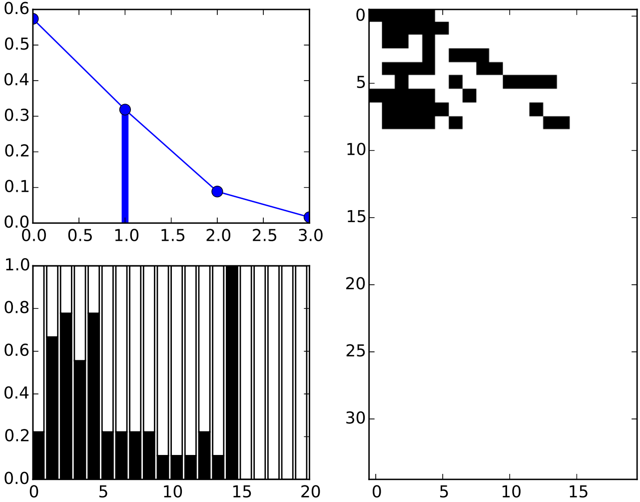

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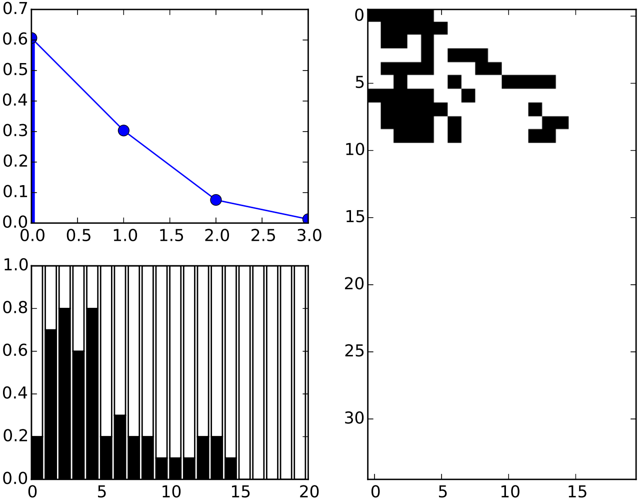

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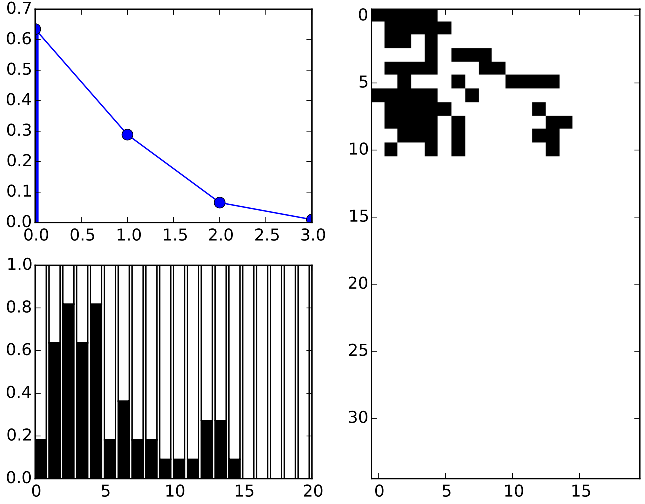

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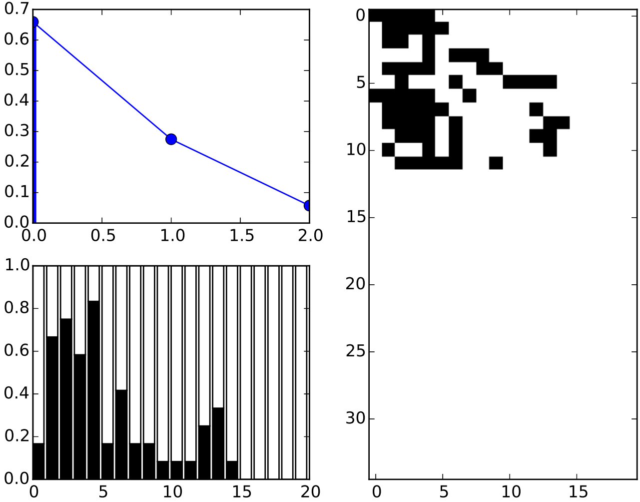

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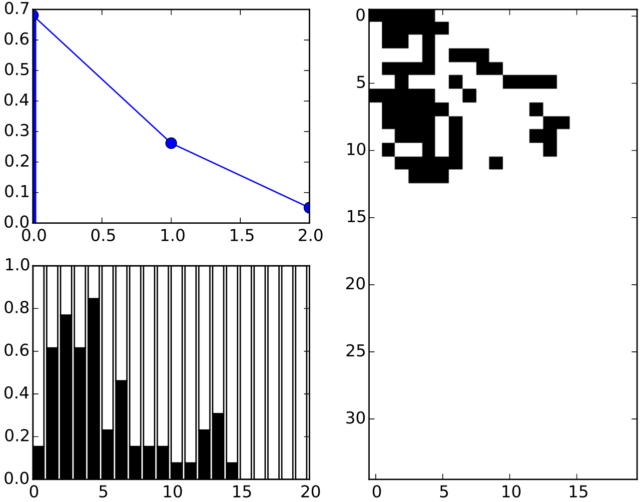

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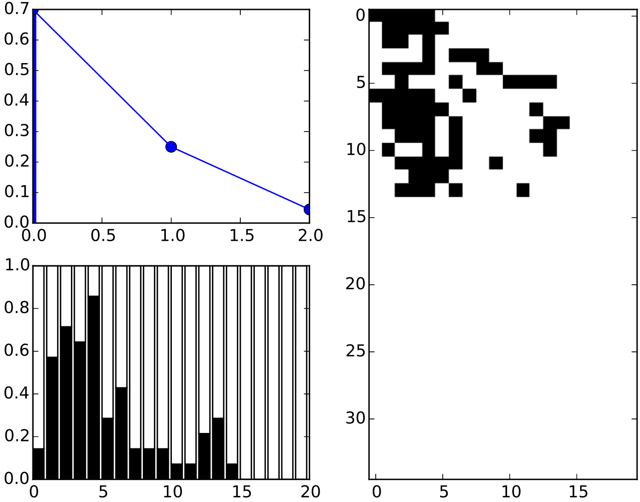

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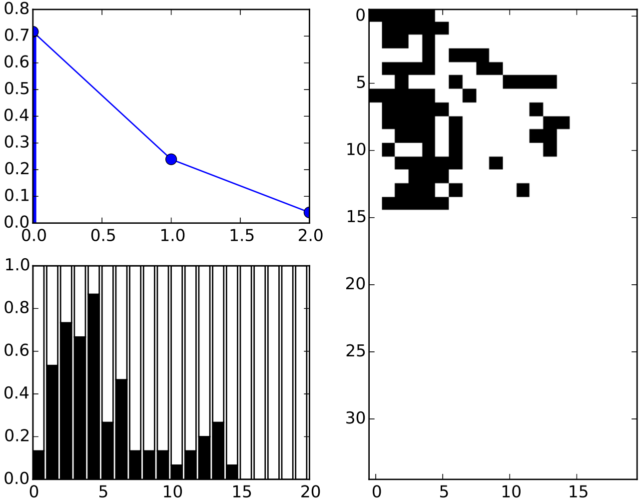

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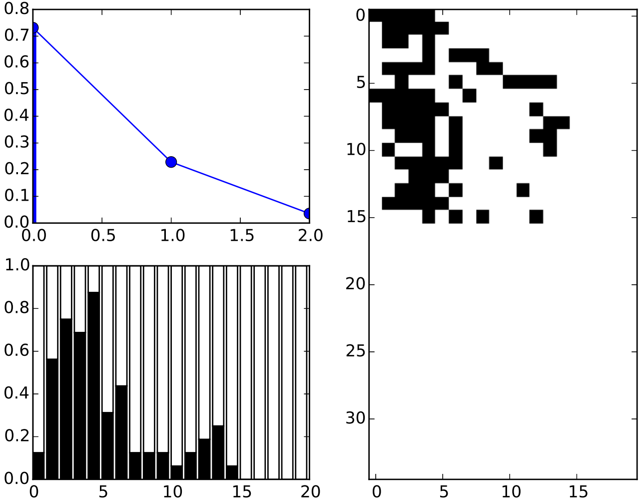

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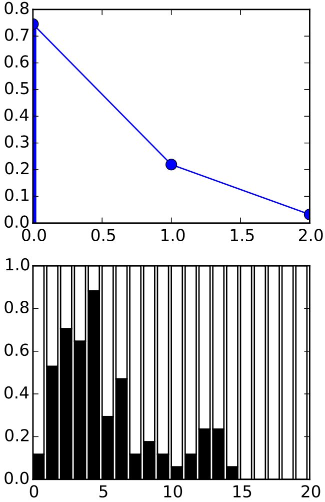

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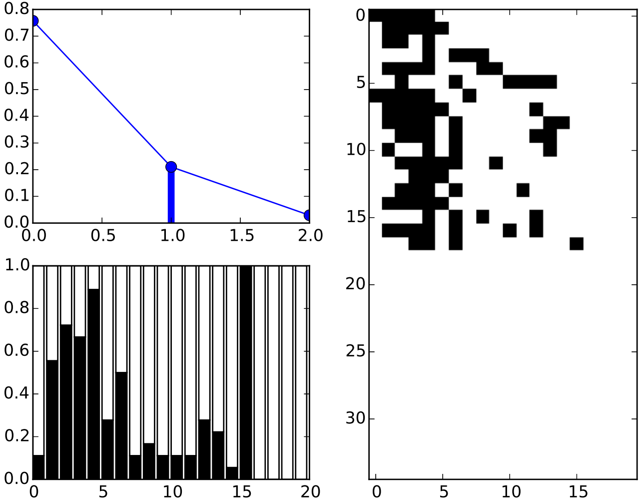

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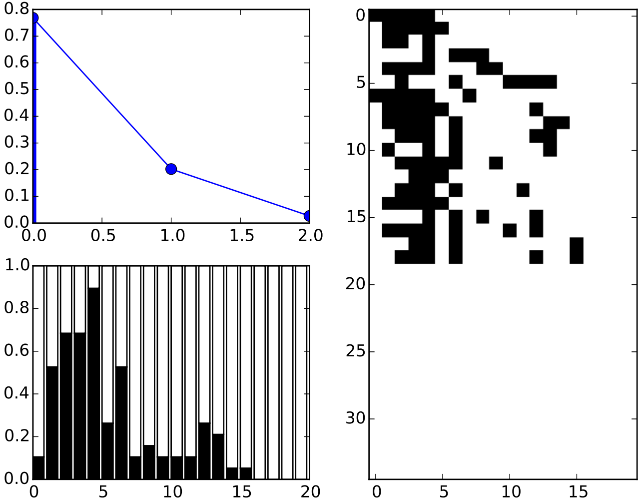

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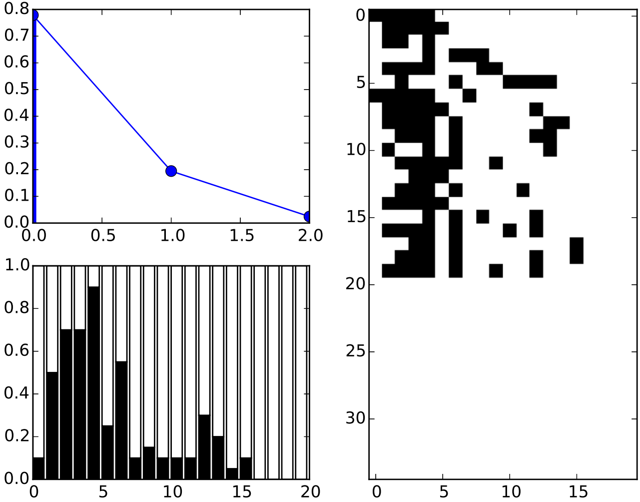

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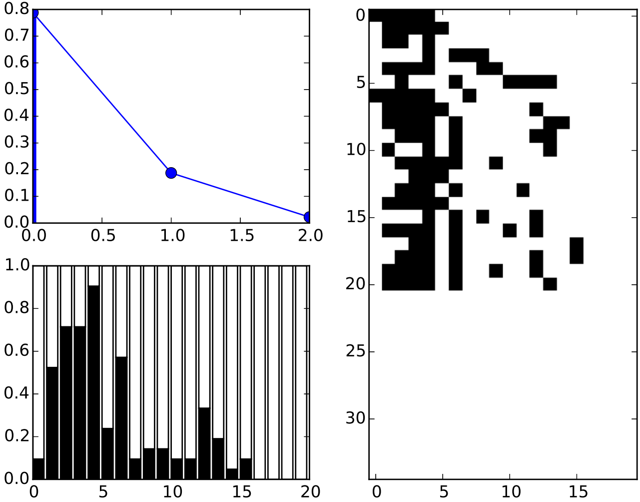

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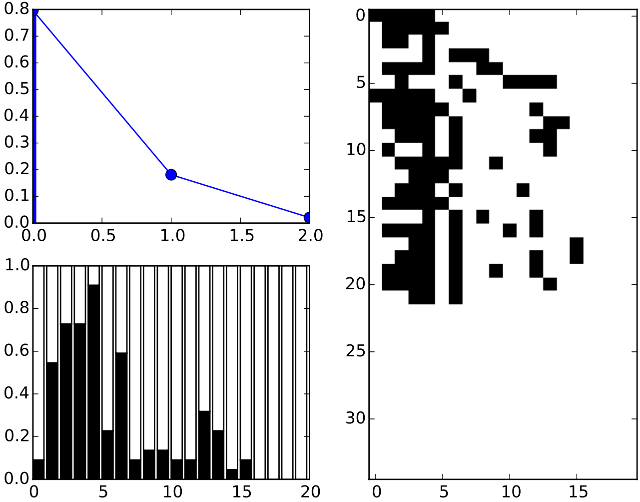

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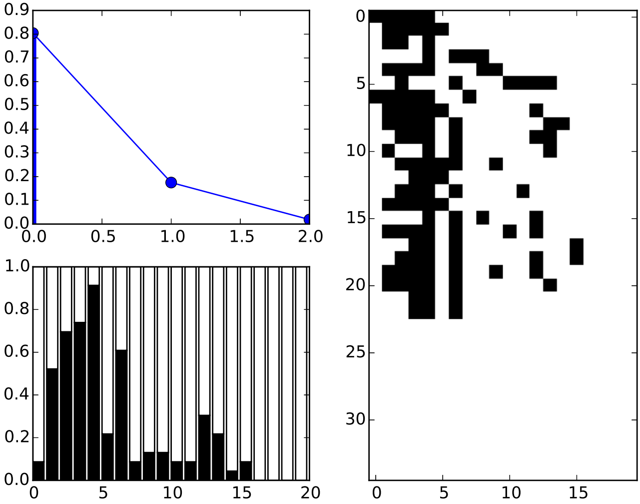

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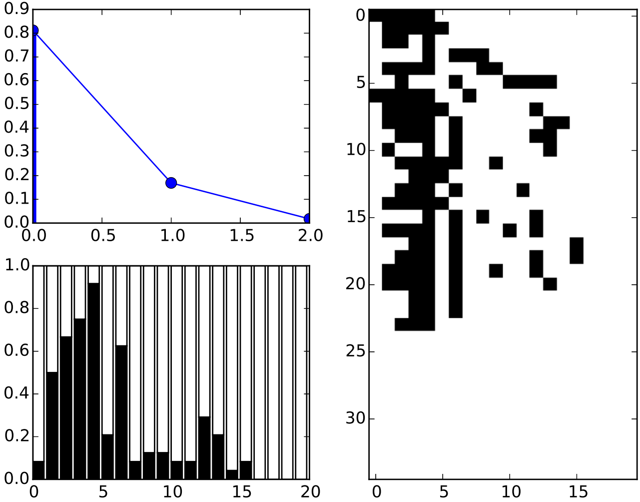

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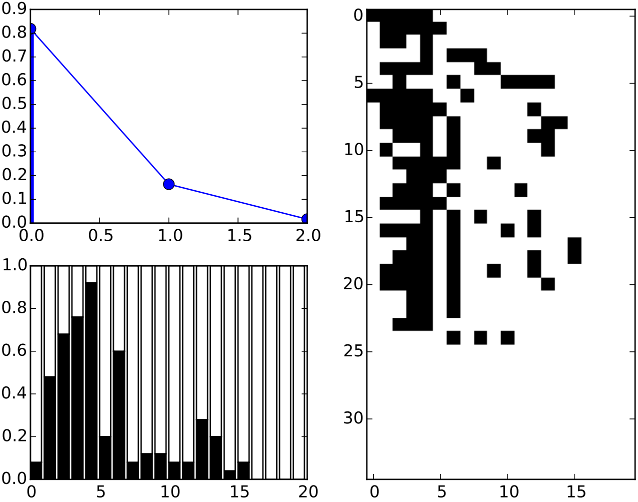

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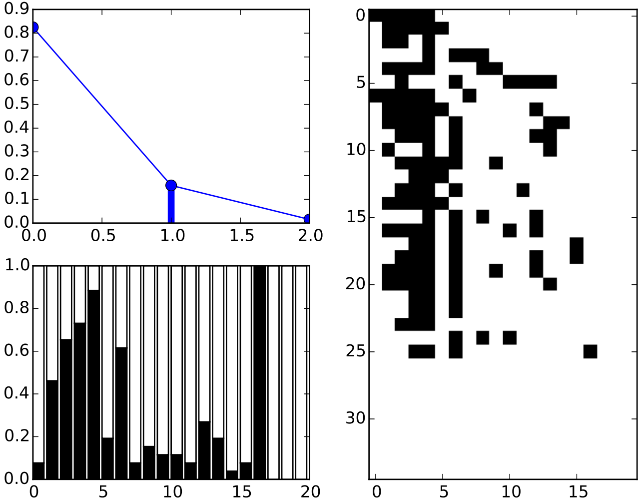

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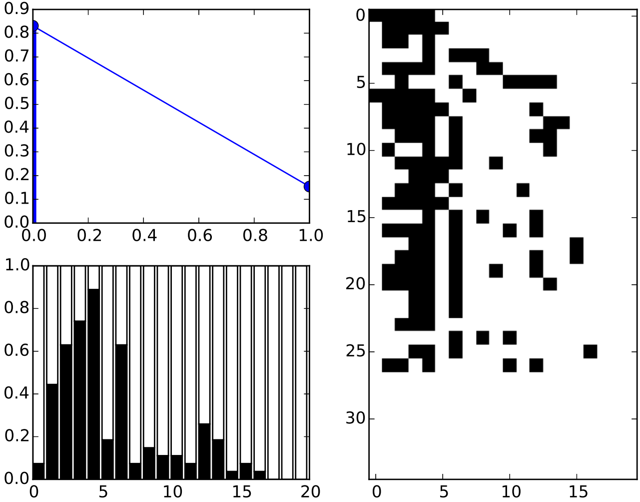

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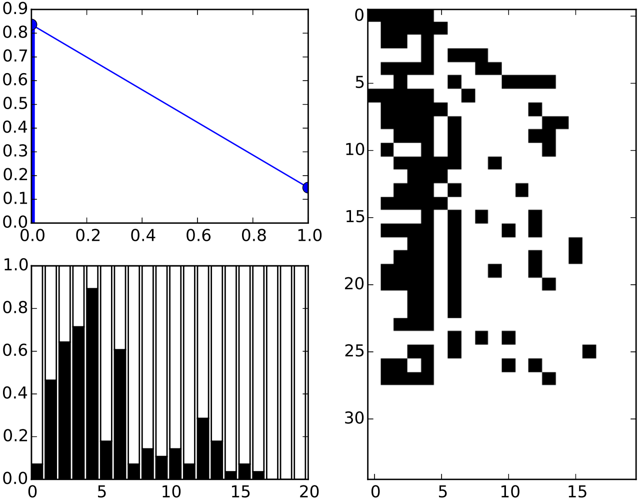

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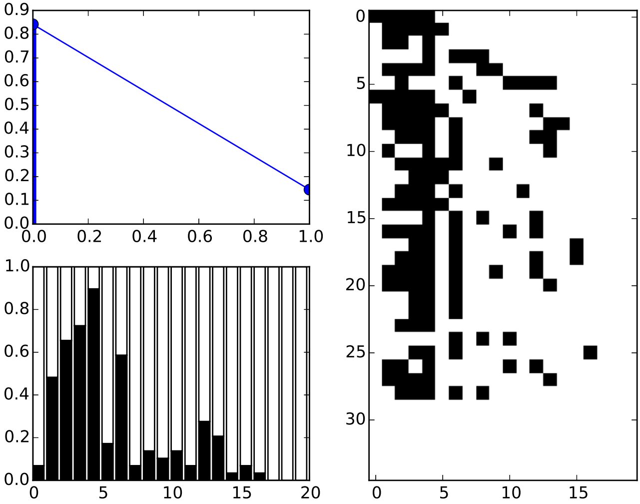

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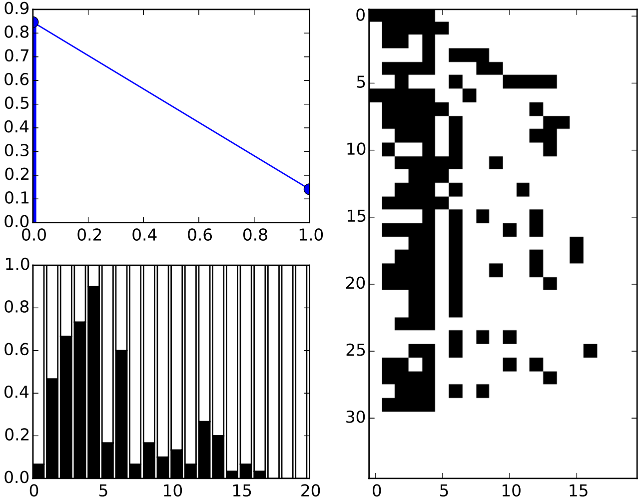

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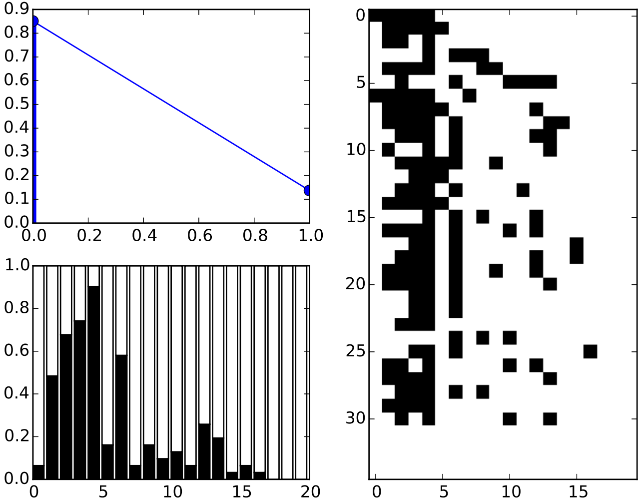

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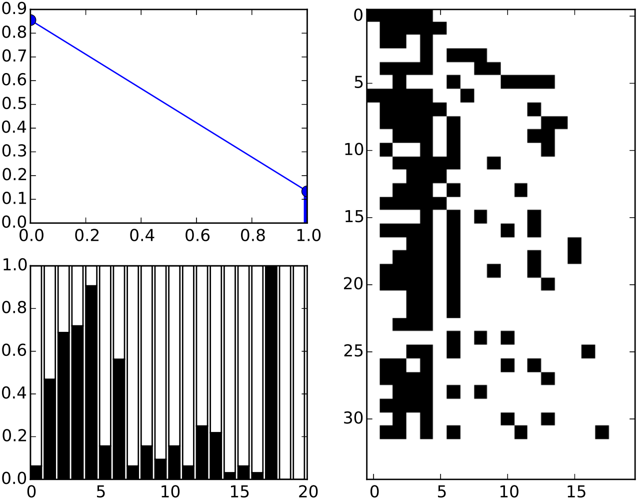

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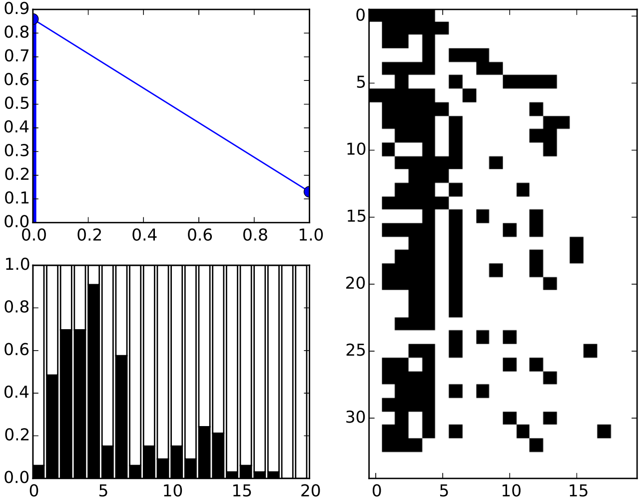

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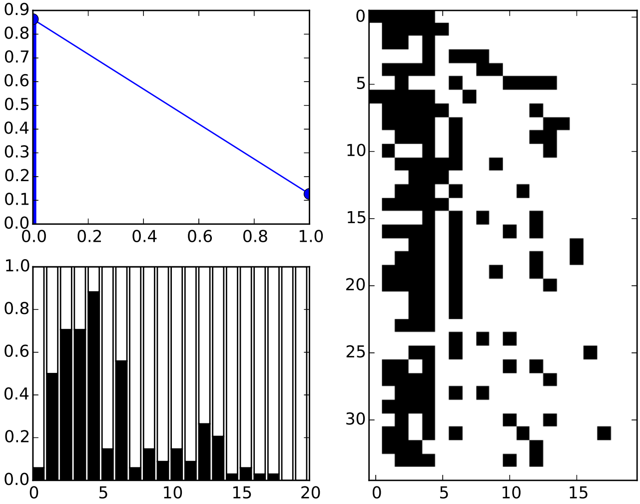
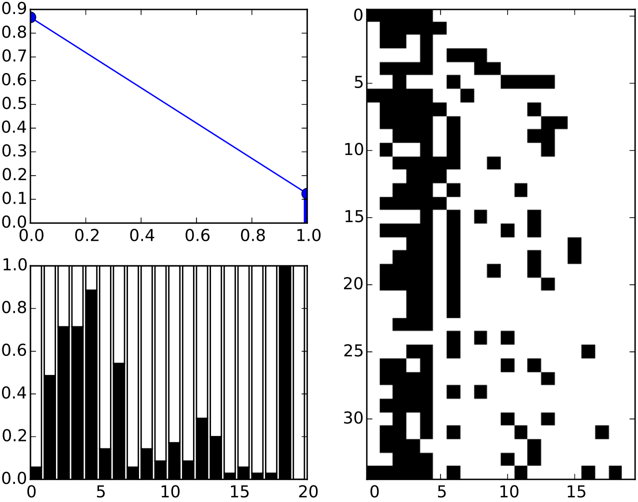


IBP Sampling

$\alpha = 5$



Gibbs Sampler

To sample, we need: $P(z_{i,k} = 1 | Z_{-i,k})$

Finite: $P(z_{i,k} = 1 | Z_{-i,k}) = \frac{n_{-i,k} + \frac{\alpha}{K}}{N + \frac{\alpha}{K}}$

Infinite: (by limit or IBP) $P(z_{i,k} = 1 | Z_{-i,k}) = \frac{n_{-i,k}}{N}$ new features: $\frac{\alpha}{N}$

Algorithm for $Z \sim P(Z)$:

- start with arbitrary binary matrix
- iterate through rows:
  - if $m_{-i,k} > 0$ set $z_{i,k} = 1$ by above
  - else, delete column k
  - add $\frac{\alpha}{N}$ new features

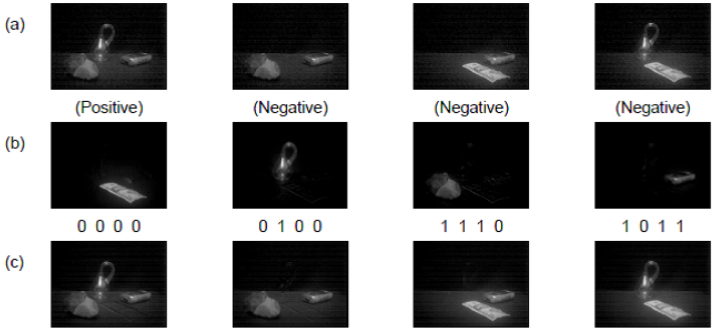
This converges to a matrix drawn from $P(Z)$

Sampling the Posterior

The real target is $P(Z|X)$
Full conditional: $P(z_{i,k} = 1|Z_{-i,k}, X) \propto P(X|Z)P(z_{i,k} = 1|Z_{-i,k})$
Algorithm:

- start with arbitrary binary matrix
- iterate through rows:
  - if $m_{-i,k} > 0$ set $z_{i,k} = 1$ *incorporating the likelihood*
  - else, delete column k
  - add new columns with prior $\frac{\alpha}{N}$ and $P(X|Z)$ likeilihood

Example Application



(a) 4 sample images from 100 (b) posterior mean of the weights of the four most frequent features, with signs (c) reconstructions of images in (a) from model with codes

Summary

- Latent feature allocation allows each sample to belong to multiple groups
- Beta prior on bernouli draws, to construct a binary matrix
- Indian Buffet Process is a generative process for the matrix marginal
- IBP yields a Gibbs Sampler
- (note) There is a stick breaking scheme... it yields variational inference

Conclusion

Bayesian nonparametrics allow distributions without *fixed* parameters

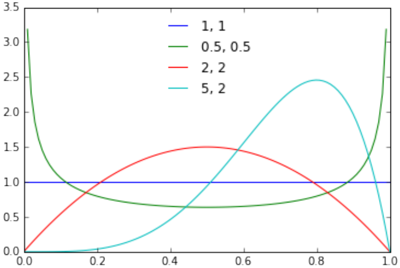
Food Metaphors explain the marginals of the categorical (CRP) or Bernouli (IBP) distributions

Food Metaphors yield Gibbs Samplers

Stick breaking metaphors yield variational inference

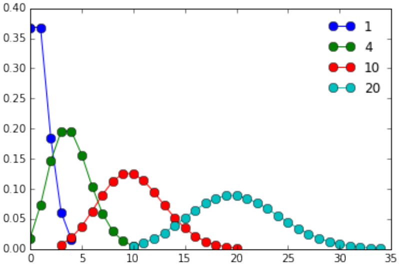
Beta Distribution

$$\rho|\alpha = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \rho^{\alpha_1 - 1}(1 - \rho)^{\alpha_2 - 1}$$



Poisson Distribution

$$k|\lambda = \frac{\lambda^k}{k!}\exp -\lambda$$



Binomial

$$p(\sum_{k=1}^{K} z_{1,k} = k) = \binom{K}{k} \frac{\alpha}{K}^{k} (1 - \frac{\alpha}{K})^{K-k}$$

marginal